20. HOMOGENEOUS DIFFERENTIAL EQUATIONS

HOMOGENEOUS FUNCTION *A function $f(x, y)$ in x and y is said to be a homogeneous function of degree n, if the degree of each term is n.*

Examples (i) $f(x, y) = (x^2 + y^2 - xy)$ is a homogeneous function of degree 2.

(ii) $g(x, y) = (x^3 - 3xy^2 + 3x^2y + y^3)$ is a homogeneous function of degree 3.

In general, a homogeneous function $f(x, y)$ of degree n is expressible as

$$f(x, y) = x^n f\left(\frac{y}{x}\right).$$

HOMOGENEOUS DIFFERENTIAL EQUATION *An equation of the form* $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$, *where both $f(x, y)$ and $g(x, y)$ are homogeneous functions of degree n, is called a homogeneous differential equation of order* 1 *and degree* 1.

Such a differential equation may be expressed in the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

For example, $\frac{dy}{dx} = \frac{x^2 - y^2}{xy} = \frac{f(x, y)}{g(x, y)}$, where $f(x, y)$ and $g(x, y)$ are homogeneous functions of degree 2 each. So, it is a homogeneous differential equation of order 1 and degree 1.

We may write, $\frac{dy}{dx} = \frac{x^2 - y^2}{xy} = \frac{\left\{1 - \left(\frac{y}{x}\right)^2\right\}}{\left\{\frac{y}{x}\right\}}$ [on dividing Nr and Dr by x^2]

Thus, $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

Method of Solving a Homogeneous Differential Equation

Let $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$ be a homogeneous differential equation of order 1 and degree 1.

Putting $y = vx$ and $\frac{dy}{dx} = \left(v + x\frac{dv}{dx}\right)$ in the given equation, we get

$$v + x\frac{dv}{dx} = F(v)$$

$$\Rightarrow \quad \frac{dv}{[F(v)-v]}=\frac{dx}{x}$$

$$\Rightarrow \quad \int\frac{dv}{[F(v)-v)]}=\int\frac{dx}{x}$$

$$\Rightarrow \quad \int\frac{dv}{[F(v)-v]}=\log|x|+C.$$

Now, replace v by (y/x) to obtain the required solution.

NOTE If the differential equation is of the form $\frac{dx}{dy}=f\left(\frac{x}{y}\right)$ then we put $x=vy$ and proceed in a manner similar to as above.

SOLVED EXAMPLES

EXAMPLE 1 *Show that the differential equation* $2x^2\frac{dy}{dx}-2xy+y^2=0$ *is homogeneous and solve it.* **[CBSE 2012]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx}=\frac{2xy-y^2}{2x^2}. \qquad \ldots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by $2x^2$, we get

$$\frac{dy}{dx}=\left\{\frac{y}{x}-\frac{1}{2}\left(\frac{y}{x}\right)^2\right\}=f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y=vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$ in (i), we get

$$v+x\frac{dv}{dx}=\frac{2vx^2-v^2x^2}{2x^2}$$

$$\Rightarrow \quad v+x\frac{dv}{dx}=\left(v-\frac{1}{2}v^2\right)$$

$$\Rightarrow \quad x\frac{dv}{dx}=-\frac{1}{2}v^2 \;\Rightarrow\; \frac{dv}{v^2}+\frac{1}{2}\frac{dx}{x}=0. \qquad \ldots \text{(ii)}$$

On integrating (ii), we get

$$\int\frac{dv}{v^2}+\frac{1}{2}\int\frac{1}{x}dx=C, \text{ where } C \text{ is an arbitrary constant}$$

$$\Rightarrow \quad \frac{-1}{v}+\frac{1}{2}\log|x|=C$$

$$\Rightarrow \quad \frac{-x}{y}+\frac{1}{2}\log|x|=C, \text{ which is the required solution.}$$

EXAMPLE 2 *Show that the differential equation* $\frac{dy}{dx}=\frac{y-x}{y+x}$ *is homogeneous and solve it.* **[CBSE 2004]**

SOLUTION The given differential equation is

$$\frac{dy}{dx}=\frac{y-x}{y+x}. \quad \ldots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by x, we get

$$\frac{dy}{dx}=\left\{\frac{\frac{y}{x}-1}{\frac{y}{x}+1}\right\}=f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y=vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$ in (i), we get

$$v+x\frac{dv}{dx}=\frac{vx-x}{vx+x}$$

$$\Rightarrow \quad v+x\frac{dv}{dx}=\frac{v-1}{v+1}$$

$$\Rightarrow \quad x\frac{dv}{dx}=\left(\frac{v-1}{v+1}-v\right)$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{-(1+v^2)}{(1+v)}$$

$$\Rightarrow \quad \frac{(1+v)}{(1+v^2)}dv=\frac{-1}{x}dx$$

$$\Rightarrow \quad \int\frac{(1+v)}{(1+v^2)}dv=-\int\frac{dx}{x}$$

$$\Rightarrow \quad \int\frac{1}{(1+v^2)}dv+\frac{1}{2}\int\frac{2v}{(1+v^2)}dv=-\int\frac{dx}{x}$$

$$\Rightarrow \quad \tan^{-1}v+\frac{1}{2}\log|1+v^2|=-\log|x|+C$$

$$\Rightarrow \quad \tan^{-1}v+\log\left|x\sqrt{1+v^2}\right|=C$$

$$\Rightarrow \quad \tan^{-1}\frac{y}{x}+\log\left|\sqrt{x^2+y^2}\right|=C \quad [\text{putting } v=\frac{y}{x}]$$

$$\Rightarrow \quad \tan^{-1}\frac{y}{x}+\frac{1}{2}\log(x^2+y^2)=C, \text{ which is the required solution.}$$

EXAMPLE 3 *Show that the differential equation* $x\frac{dy}{dx}-y=\sqrt{x^2+y^2}$ *is homogeneous and solve it.* **[CBSE 2005, '07, '11]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}. \qquad \ldots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by x, we get

$$\frac{dy}{dx} = \left\{\frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2}\right\} = f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2x^2}}{x} = v + \sqrt{1 + v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \sqrt{1 + v^2}$$

$$\Rightarrow \quad \frac{dv}{\sqrt{1 + v^2}} = \frac{1}{x}dx$$

$$\Rightarrow \quad \int \frac{dv}{(1 + v^2)} = \int \frac{dx}{x}$$

$$\Rightarrow \quad \log\left|v + \sqrt{1 + v^2}\right| = \log|x| + \log|C_1|,$$

where C_1 is an arbitrary constant

$$\Rightarrow \quad \log\left|\frac{v + \sqrt{1 + v^2}}{x}\right| = \log|C_1|$$

$$\Rightarrow \quad \frac{v + \sqrt{1 + v^2}}{x} = \pm C_1 = C \text{ (say)}$$

$$\Rightarrow \quad v + \sqrt{1 + v^2} = Cx$$

$$\Rightarrow \quad y + \sqrt{x^2 + y^2} = Cx^2, \text{ which is the required solution } \left[\because \ v = \frac{y}{x}\right].$$

EXAMPLE 4 *Show that the differential equation* $(x\sqrt{x^2 + y^2} - y^2)dx + xydy = 0$ *is homogeneous and solve it.*

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{y^2 - x\sqrt{x^2 + y^2}}{xy}. \qquad \ldots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by x^2, we get

$$\frac{dy}{dx} = \left\{\frac{\left(\frac{y}{x}\right)^2 - \sqrt{1 + \left(\frac{y}{x}\right)^2}}{\left(\frac{y}{x}\right)}\right\} = f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x\sqrt{x^2 + v^2x^2}}{vx^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \left(\frac{v^2 - \sqrt{1+v^2}}{v} - v\right)$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-\sqrt{1+v^2}}{v}$$

$$\Rightarrow \quad \int \frac{v}{\sqrt{1+v^2}}dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \sqrt{1+v^2} = -\log|x| + C$$

$$\Rightarrow \quad \sqrt{x^2+y^2} + x\log|x| = Cx, \text{ which is the required solution.}$$

EXAMPLE 5 *Show that the differential equation* $(x^2 + xy)dy = (x^2 + y^2)dx$ *is homogeneous and solve it.* **[CBSE 2005]**

SOLUTION The given differential equation is

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy}. \qquad \ldots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by x^2, we get

$$\frac{dy}{dx} = \frac{\left\{1 + \left(\frac{y}{x}\right)^2\right\}}{\left\{1 + \frac{y}{x}\right\}} = f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{x^2 + vx^2} = \frac{1+v^2}{1+v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{(1+v^2)}{(1+v)} - v = \frac{1+v^2-v-v^2}{(1+v)} = \frac{(1-v)}{(1+v)}$$

$$\Rightarrow \quad \frac{(1+v)}{(1-v)}dv = \frac{1}{x}dx$$

$$\Rightarrow \quad \int \frac{(1+v)}{(1-v)}dv = \int \frac{1}{x}dx \quad \text{[on integrating both sides]}$$

$$\Rightarrow \quad \int \frac{\{2-(1-v)\}}{(1-v)}dv = \int \frac{1}{x}dx$$

$$\Rightarrow \int\left\{\frac{2}{(1-v)}-1\right\}dv=\int\frac{1}{x}dx$$

$$\Rightarrow -2\log|1-v|-v=\log|x|+\log|C|,$$

where C is an arbitrary constant

$$\Rightarrow \log|x|+\log|C|+2\log|1-v|=-v$$

$$\Rightarrow \log|Cx(1-v)^2|=-v$$

$$\Rightarrow |Cx(1-v)^2|=e^{-v}$$

$$\Rightarrow \left|Cx\left(1-\frac{y}{x}\right)^2\right|=e^{-y/x} \quad \left[\because\ v=\frac{y}{x}\right]$$

$\Rightarrow |C|(x-y)^2=|x|e^{-y/x}$, which is the required solution.

EXAMPLE 6 *Show that the differential equation* $y^2dx+(x^2-xy+y^2)dy=0$ *is homogeneous and solve it.*

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx}=\frac{-y^2}{(x^2-xy+y^2)}. \qquad \ldots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by x^2, we get

$$\frac{dy}{dx}=\frac{-\left(\frac{y}{x}\right)^2}{\left\{1-\frac{y}{x}+\left(\frac{y}{x}\right)^2\right\}}=f\left(\frac{y}{x}\right).$$

Thus, the given differential equation is homogeneous.

Putting $y=vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$ in (i), we get

$$v+x\frac{dv}{dx}=\frac{-v^2}{(1-v+v^2)}$$

$$\Rightarrow x\frac{dv}{dx}=-\left[\frac{v^2}{(1-v+v^2)}+v\right]$$

$$\Rightarrow x\frac{dv}{dx}=\frac{-(v+v^3)}{(1-v+v^2)}$$

$$\Rightarrow \frac{(1-v+v^2)}{v(1+v^2)}dv=-\frac{1}{x}dx$$

$$\Rightarrow \int\frac{(1+v^2)-v}{v(1+v^2)}dv=-\int\frac{dx}{x}$$

$$\Rightarrow \int\frac{dv}{v}-\int\frac{dv}{(1+v^2)}+\int\frac{dx}{x}=\log C$$

$$\Rightarrow \log|v|-\tan^{-1}v+\log|x|=\log C$$

$$\Rightarrow \tan^{-1}v=\log\frac{|vx|}{C}$$

$$\Rightarrow \quad \tan^{-1}\frac{y}{x} = \log\left(\frac{|y|}{C}\right) \quad \left[\because \ v = \frac{y}{x}\right]$$

$$\Rightarrow \quad \frac{|y|}{C} = e^{\tan^{-1}(y/x)}$$

$$\Rightarrow \quad |y| = Ce^{\tan^{-1}(y/x)}, \text{ which is the required solution.}$$

EXAMPLE 7 *Show that the differential equation* $x^2\left(\frac{dy}{dx}\right) = (x^2 - 2y^2 + xy)$ *is homogeneous and solve it.*

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{(x^2 - 2y^2 + xy)}{x^2}. \qquad \ldots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by x^2, we get

$$\frac{dy}{dx} = \left\{1 - 2\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)\right\} = f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{x^2 - 2v^2x^2 + vx^2}{x^2}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = 1 - 2v^2 + v$$

$$\Rightarrow \quad x\frac{dv}{dx} = 1 - 2v^2$$

$$\Rightarrow \quad \frac{dv}{(1-2v^2)} = \frac{dx}{x}$$

$$\Rightarrow \quad \int\frac{dv}{(1-2v^2)} = \int\frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\int\frac{dv}{\left(\frac{1}{2} - v^2\right)} = \int\frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\int\frac{dv}{\left\{\left(\frac{1}{\sqrt{2}}\right)^2 - v^2\right\}} = \int\frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\cdot\frac{1}{\left(2\times\frac{1}{\sqrt{2}}\right)}\log\left|\frac{\frac{1}{\sqrt{2}} + v}{\frac{1}{\sqrt{2}} - v}\right| = \log|x| + C$$

$$\Rightarrow \quad \frac{1}{2\sqrt{2}}\log\left|\frac{x + \sqrt{2}y}{x - \sqrt{2}y}\right| - \log|x| = C \quad [\because \ v = \frac{y}{x}].$$

This is the required solution.

EXAMPLE 8 *Show that the differential equation* $(x^3+y^3)dy - x^2ydx = 0$ *is homogeneous and solve it.* **[CBSE 2008]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{x^2y}{(x^3+y^3)}. \qquad \ldots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by x^3, we get

$$\frac{dy}{dx} = \frac{\left(\frac{y}{x}\right)}{\left\{1+\left(\frac{y}{x}\right)^3\right\}} = f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{vx^3}{x^3+v^3x^3}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v}{1+v^3}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \left(\frac{v}{1+v^3} - v\right) = \frac{-v^4}{(1+v^3)}$$

$$\Rightarrow \quad \frac{(1+v^3)}{v^4}dv = \frac{-1}{x}dx$$

$$\Rightarrow \quad \int\left(\frac{1}{v^4}+\frac{1}{v}\right)dv = -\int\frac{1}{x}dx$$

$$\Rightarrow \quad \int\left(\frac{1}{v^4}+\frac{1}{v}\right)dv + \int\frac{1}{x}dx = C$$

$$\Rightarrow \quad \frac{-1}{3v^3} + \log|v| + \log|x| = C$$

$$\Rightarrow \quad \frac{-1}{3v^3} + \log|vx| = C$$

$$\Rightarrow \quad \frac{-x^3}{3y^3} + \log|y| = C, \text{ which is the required solution.}$$

EXAMPLE 9 *Show that the differential equation* $(y^2-x^2)dy = 3xy\,dx$ *is homogeneous and solve it.* **[CBSE 2006, '08]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{3xy}{(y^2-x^2)}. \qquad \ldots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by x^2, we get

$$\frac{dy}{dx}=\frac{3\left(\frac{y}{x}\right)}{\left\{\left(\frac{y}{x}\right)^2-1\right\}}=f\left(\frac{y}{x}\right).$$

Thus, the given differential equation is homogeneous.

Putting $y=vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$ in (i), we get

$$v+x\frac{dv}{dx}=\frac{3v}{(v^2-1)}$$

$$\Rightarrow \quad x\frac{dv}{dx}=\left\{\frac{3v}{(v^2-1)}-v\right\}=\frac{(4v-v^3)}{(v^2-1)}$$

$$\Rightarrow \quad \frac{(v^2-1)}{(4v-v^3)}dv=\frac{1}{x}dx$$

$$\Rightarrow \quad \int\frac{(v^2-1)}{v(2-v)(2+v)}dv=\int\frac{1}{x}dx. \qquad \ldots \text{(ii)}$$

Let $\frac{(v^2-1)}{v(2-v)(2+v)}=\frac{A}{v}+\frac{B}{(2-v)}+\frac{C}{(2+v)}$.

Then, $(v^2-1)\equiv A(2-v)(2+v)+Bv(2+v)+Cv(2-v)$. … (iii)

Putting $v=0$ on each side of (iii), we get $A=\frac{-1}{4}$.

Putting $v=2$ on each side of (iii), we get $B=\frac{3}{8}$.

Putting $v=-2$ on each side of (iii), we get $C=\frac{-3}{8}$.

$$\therefore \quad \frac{(v^2-1)}{v(2-v)(2+v)}=\frac{-1}{4v}+\frac{3}{8(2-v)}-\frac{3}{8(2+v)}. \qquad \ldots \text{(iv)}$$

Putting these values from (iv) in (ii), we get

$$-\frac{1}{4}\int\frac{dv}{v}+\frac{3}{8}\int\frac{dv}{(2-v)}-\frac{3}{8}\int\frac{dv}{(2+v)}=\int\frac{1}{x}dx$$

$$\Rightarrow \quad \int\frac{1}{x}dx+\frac{1}{4}\int\frac{dv}{v}+\frac{3}{8}\int\frac{-dv}{(2-v)}+\frac{3}{8}\int\frac{dv}{(2+v)}=\log|C_1|,$$

where C_1 is an arbitrary constant

$$\Rightarrow \quad \log|x|+\frac{1}{4}\log|v|+\frac{3}{8}\log|2-v|+\frac{3}{8}\log|2+v|=\log|C_1|$$

$$\Rightarrow \quad 8\log|x|+2\log|v|+3\log|2-v|+3\log|2+v|=8\log|C_1|$$

$$\Rightarrow \quad \log|x^8v^2(2-v)^3(2+v)^3|=\log\{(C_1)^8\}$$

$$\Rightarrow \quad |x^8v^2(2-v)^3(2+v)^3|=C_1^8=C \text{ (say)}$$

$$\Rightarrow \quad x^8\frac{y^2}{x^2}\left(2-\frac{y}{x}\right)^3\left(2+\frac{y}{x}\right)^3=C$$

$\Rightarrow \quad y^2(2x-y)^3(2x+y)^3 = C$, where C is an arbitrary constant.

$\Rightarrow \quad y^2(4x^2-y^2)^3 = C$, which is the required solution.

EXAMPLE 10 *Show that the differential equation* $(x^3-3xy^2)dx = (y^3-3x^2y)dy$ *is homogeneous and solve it.*

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{x^3-3xy^2}{y^3-3x^2y}. \qquad \ldots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by x^3, we get

$$\frac{dy}{dx} = \frac{1-3\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)^3-3\left(\frac{y}{x}\right)} = f\left(\frac{y}{x}\right).$$

Thus, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{x^3-3v^2x^3}{v^3x^3-3vx^3}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{1-3v^2}{v^3-3v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \left(\frac{1-3v^2}{v^3-3v} - v\right)$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{(1-v^4)}{(v^3-3v)}$$

$$\Rightarrow \quad \frac{(3v-v^3)}{(v^4-1)}dv = \frac{dx}{x}$$

$$\Rightarrow \quad \int\frac{(3v-v^3)}{(v^4-1)}dv = \int\frac{dx}{x}. \qquad \ldots \text{(ii)}$$

Let $\frac{(3v-v^3)}{(v^4-1)} = \frac{A}{(v-1)} + \frac{B}{(v+1)} + \frac{Cv+D}{(v^2+1)}$. Then,

$$(3v-v^3) \equiv A(v+1)(v^2+1) + B(v-1)(v^2+1) + (Cv+D)(v-1)(v+1). \qquad \ldots \text{(iii)}$$

Putting $v = 1$ on each side of (iii), we get $A = \frac{1}{2}$.

Putting $v = -1$ on each side of (iii), we get $B = \frac{1}{2}$.

Comparing coefficients of v^3 on both sides of (iii), we get

$$A + B + C = -1 \;\Rightarrow\; \frac{1}{2} + \frac{1}{2} + C = -1 \;\Rightarrow\; C = -2.$$

Comparing the independent terms on both sides of (iii), we get

$$A - B - D = 0 \Rightarrow D = (A - B) = \left(\frac{1}{2} - \frac{1}{2}\right) = 0.$$

$$\therefore \quad \frac{(3v - v^3)}{(v^4 - 1)} = \frac{1}{2(v-1)} + \frac{1}{2(v+1)} - \frac{2v}{(v^2+1)}.$$

Putting this value in (ii), we get

$$\frac{1}{2}\int \frac{dv}{(v-1)} + \frac{1}{2}\int \frac{dv}{(v+1)} - \int \frac{2v}{(v^2+1)}dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\log|v-1| + \frac{1}{2}\log|v+1| - \log|v^2+1| = \log|x| + \log|C|,$$

where C is an arbitrary constant

$$\Rightarrow \quad \log|v-1| + \log|v+1| - 2\log|v^2+1| = 2\log|x| + 2\log|C|$$

$$\Rightarrow \quad \log\left|\frac{(v-1)(v+1)}{(v^2+1)^2}\right| = \log|C^2x^2| \Rightarrow \log\left|\frac{(v^2-1)}{(v^2+1)^2}\right| = \log|C^2x^2|$$

$$\Rightarrow \quad (y^2 - x^2) = C^2(y^2 + x^2)^2 \quad \left[\because \ v = \frac{y}{x}\right].$$

Hence, $(y^2 - x^2) = C^2(y^2 + x^2)^2$ is the required solution.

EXAMPLE 11 *Show that the differential equation* $(x - y)\dfrac{dy}{dx} = (x + 2y)$ *is homogeneous and solve it.* **[CBSE 2013C]**

SOLUTION The given differential equation can be expressed as

$$\frac{dy}{dx} = \frac{x + 2y}{x - y}. \qquad \ldots \text{(i)}$$

On dividing the Nr and Dr of (i) by x, we get

$$\frac{dy}{dx} = \frac{\left\{1 + 2\cdot\dfrac{y}{x}\right\}}{\left\{1 - \dfrac{y}{x}\right\}} = f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{1 + 2v}{1 - v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \left\{\frac{1 + 2v}{1 - v} - v\right\} = \frac{(1 + 2v) - v + v^2}{(1 - v)} = \frac{v^2 + v + 1}{(1 - v)}$$

$$\Rightarrow \quad \frac{(v-1)}{(v^2 + v + 1)}dv = -\frac{1}{x}dx \Rightarrow \int \frac{(v-1)}{(v^2 + v + 1)}dv = -\int \frac{1}{x}dx. \qquad \ldots \text{(ii)}$$

Let $(v - 1) = A \cdot \dfrac{d}{dv}(v^2 + v + 1) + B.$

Then, $(v-1) = A(2v+1) + B.$ … (iii)

Comparing coefficients of like powers of v, we get

$$(2A = 1 \text{ and } A + B = -1) \Rightarrow \left(A = \frac{1}{2} \text{ and } B = \frac{-3}{2}\right).$$

Putting $(v-1) = \frac{1}{2}(2v+1) - \frac{3}{2}$ in (ii), we get

$$\int \frac{\frac{1}{2}(2v+1) - \frac{3}{2}}{(v^2+v+1)} dv = -\log|x| + C$$

$$\Rightarrow \frac{1}{2}\int \frac{(2v+1)}{(v^2+v+1)} dv - \frac{3}{2}\int \frac{dv}{(v^2+v+1)} = -\log|x| + C$$

$$\Rightarrow \frac{1}{2}\log|v^2+v+1| - \frac{3}{2}\int \frac{1}{\left(v+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dv = -\log|x| + C$$

$$\Rightarrow \frac{1}{2}\log|v^2+v+1| - \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) = -\log|x| + C$$

$$\Rightarrow \frac{1}{2}\log|v^2+v+1| + \frac{1}{2}\log x^2 = \sqrt{3}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) + C$$

$$\Rightarrow \frac{1}{2}\log\left|\frac{y^2}{x^2} + \frac{y}{x} + 1\right| + \frac{1}{2}\log x^2 = \sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) + C$$

$$\Rightarrow \frac{1}{2}\log\left\{\frac{|(y^2+xy+x^2)|}{x^2}\cdot x^2\right\} = \sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) + C$$

$$\Rightarrow \log|x^2+xy+y^2| = 2\sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right) + C,$$

which is the required solution.

EXAMPLE 12 *Show that the differential equation* $x\frac{dy}{dx} = y - x\tan\frac{y}{x}$ *is homogeneous and solve it.* **[CBSE 2006C, '09]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{y}{x} - \tan\frac{y}{x}. \quad \text{… (i)}$$

This is of the form, $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = v - \tan v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\tan v$$

$$\Rightarrow \quad \frac{dv}{\tan v} = -\frac{dx}{x}$$

$$\Rightarrow \quad \int \cot v\, dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{\cos v}{\sin v} dv + \int \frac{1}{x} dx = \log |C_1|, \text{ where } C_1 \text{ is an arbitrary constant}$$

$$\Rightarrow \quad \log |\sin v| + \log |x| = \log |C_1|$$

$$\Rightarrow \quad \log |x \sin v| = \log |C_1|$$

$$\Rightarrow \quad x \sin v = \pm C_1 = C \text{ (say)}$$

$$\Rightarrow \quad x \sin \frac{y}{x} = C, \text{ which is the required solution.}$$

EXAMPLE 13 *Show that the differential equation*

$$\left(x \cos \frac{y}{x}\right)(y\,dx + x\,dy) = \left(y \sin \frac{y}{x}\right)(x\,dy - y\,dx)$$

is homogeneous and solve it. **[CBSE 2010, '13C]**

SOLUTION The given differential equation may be written as

$$\left(xy \cos \frac{y}{x} + y^2 \sin \frac{y}{x}\right) dx = \left(xy \sin \frac{y}{x} - x^2 \cos \frac{y}{x}\right) dy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\left\{xy \cos \frac{y}{x} + y^2 \sin \frac{y}{x}\right\}}{\left\{xy \sin \frac{y}{x} - x^2 \cos \frac{y}{x}\right\}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(y/x)\cos (y/x) + (y/x)^2 \sin (y/x)}{(y/x)\sin (y/x) - \cos (y/x)} = f\left(\frac{y}{x}\right) \qquad \ldots \text{(i)}$$

[dividing Nr and Dr by x^2]

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{(v \cos v + v^2 \sin v)}{(v \sin v - \cos v)}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \left\{\frac{(v \cos v + v^2 \sin v)}{(v \sin v - \cos v)} - v\right\}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2v \cos v}{(v \sin v - \cos v)}$$

$$\Rightarrow \quad \int \frac{(v \sin v - \cos v)}{v \cos v} dv = \int \frac{2}{x} dx$$

$$\Rightarrow \int \tan v\, dv - \int \frac{dv}{v} = \int \frac{2}{x} dx$$

$$\Rightarrow -\log|\cos v| - \log|v| - 2\log|x| = \text{constant}$$

$$\Rightarrow \log|\cos v| + \log|v| + 2\log|x| = \log|C_1|,$$

where C_1 is an arbitrary constant

$$\Rightarrow \log|x^2 v \cos v| = \log|C_1|$$

$$\Rightarrow x^2 v \cos v = \pm C_1 = C \text{ (say)}$$

$$\Rightarrow xy\cos\frac{y}{x} = C, \text{ which is the required solution } \left[\because\ v = \frac{y}{x}\right].$$

EXAMPLE 14 *Show that the differential equation*

$$xy\log\left(\frac{y}{x}\right)dx + \left\{y^2 - x^2\log\left(\frac{y}{x}\right)\right\}dy = 0 \text{ is homogeneous and solve it.}$$

[CBSE 2010C]

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{xy\log\left(\frac{y}{x}\right)}{\left\{x^2\log\left(\frac{y}{x}\right) - y^2\right\}}. \qquad \ldots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by x^2, we get

$$\frac{dy}{dx} = \frac{\left(\frac{y}{x}\right)\log\left(\frac{y}{x}\right)}{\left\{\log\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2\right\}} = f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{vx^2\log v}{x^2(\log v - v^2)} = \frac{v\log v}{(\log v - v^2)}$$

$$\Rightarrow x\frac{dv}{dx} = \left\{\frac{v\log v}{(\log v - v^2)} - v\right\} = \frac{v^3}{(\log v - v^2)}$$

$$\Rightarrow \frac{(\log v - v^2)}{v^3}dv = \frac{1}{x}dx$$

$$\Rightarrow \int\left\{\frac{\log v}{v^3} - \frac{1}{v}\right\}dv = \int\frac{1}{x}dx \quad \text{[integrating both sides]}$$

$$\Rightarrow \int \underset{\text{I}}{(\log v)}\,\underset{\text{II}}{v^{-3}}\,dv - \int\frac{1}{v}dv = \int\frac{1}{x}dx + C_1,$$

where C_1 is an arbitrary constant

$$\Rightarrow (\log v)\cdot\left(\frac{v^{-2}}{-2}\right) - \int\frac{1}{v}\cdot\left(\frac{v^{-2}}{-2}\right)dv - \log|v| = \log|x| + C_1$$

[integrating by parts]

$$\Rightarrow \quad \frac{-\log v}{2v^2} + \frac{1}{2}\int v^{-3}dv - \log|v| = \log|x| + C_1$$

$$\Rightarrow \quad \frac{-\log v}{2v^2} - \frac{1}{4v^2} - \log|v| = \log|x| + C_1$$

$$\Rightarrow \quad \frac{\log v}{2v^2} + \frac{1}{4v^2} + \log|v| + \log|x| = -C_1 = C \text{ (say)}$$

$$\Rightarrow \quad \frac{\log\left|\frac{y}{x}\right|}{2\left(\frac{y^2}{x^2}\right)} + \frac{x^2}{4y^2} + \log|y| = C \quad [\because \quad v = \frac{y}{x}]$$

$$\Rightarrow \quad 2x^2 \log\left|\frac{y}{x}\right| + x^2 + 4y^2 \log|y| = 4Cy^2$$

$$\Rightarrow \quad x^2\left(2\log\left|\frac{y}{x}\right| + 1\right) + 4y^2 \log|y| = 4Cy^2,$$

which is the required solution.

EXAMPLE 15 *Show that the differential equation* $x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) + x - y\sin\left(\frac{y}{x}\right) = 0$ *is homogeneous. Find the particular solution of this differential equation, given that* $x = 1$ *when* $y = \frac{\pi}{2}$. **[CBSE 2013]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{(y/x)\sin(y/x) - 1}{\sin(y/x)} = f\left(\frac{y}{x}\right). \qquad \ldots \text{(i)}$$

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \left\{\frac{v\sin v - 1}{\sin v} - v\right\} = \frac{-1}{\sin v}$$

$$\Rightarrow \quad \sin v\, dv = \frac{-1}{x}dx$$

$$\Rightarrow \quad \int \sin v\, dv = -\int \frac{1}{x}dx \quad \text{[on integrating both sides]}$$

$$\Rightarrow \quad -\cos v = -\log|x| + C, \text{ where } C \text{ is an arbitrary constant}$$

$$\Rightarrow \quad \log|x| - \cos\left(\frac{y}{x}\right) = C \qquad \ldots \text{(ii)} \quad \left[\because \quad v = \frac{y}{x}\right].$$

Putting $x = 1$ and $y = \frac{\pi}{2}$ in (ii), we get $C = 0$.

Hence, $\log|x| = \cos\left(\frac{y}{x}\right)$ is the required solution.

EXAMPLE 16 *Find the particular solution of the differential equation* $(3xy + y^2)\,dx + (x^2 + xy)\,dy = 0$ *for* $x = 1$ *and* $y = 1$. **[CBSE 2004C, '07, '13C]**

SOLUTION We may write the given differential equation as

$$\frac{dy}{dx} = \frac{-(3xy + y^2)}{(x^2 + xy)}. \qquad \ldots (i)$$

On dividing the Nr and Dr of RHS of (i) by x^2, we get

$$\frac{dy}{dx} = \frac{-\left\{3\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2\right\}}{\left\{1 + \frac{y}{x}\right\}} = f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{-(3vx^2 + v^2x^2)}{(x^2 + vx^2)} = \frac{-(3v + v^2)}{(v + 1)}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \left\{\frac{-(3v + v^2)}{(v + 1)} - v\right\} = \frac{-2(v^2 + 2v)}{(v + 1)}$$

$$\Rightarrow \quad \frac{(v + 1)}{(v^2 + 2v)}dv = \frac{-2}{x}dx$$

$$\Rightarrow \quad \frac{1}{2}\int \frac{2(v + 1)}{(v^2 + 2v)}dv + \int \frac{2}{x}dx = \log|C_1|$$

$$\Rightarrow \quad \frac{1}{2}\log|v^2 + 2v| + 2\log|x| = \log|C_1|$$

$$\Rightarrow \quad \log|x^2\sqrt{v^2 + 2v}| = \log|C_1|$$

$$\Rightarrow \quad \log|x\sqrt{y^2 + 2xy}| = \log|C_1| \quad \left[\because\ v = \frac{y}{x}\right].$$

$$\Rightarrow \quad x\sqrt{y^2 + 2xy} = \pm C_1$$

$$\Rightarrow \quad x^2(y^2 + 2xy) = C, \text{ where } C = C_1^2. \qquad \ldots (ii)$$

Putting $x = 1$ and $y = 1$ in (ii), we get $C = 3$.

$\therefore \quad x^2(y^2 + 2xy) = 3$ is the required solution.

EXAMPLE 17 *Find the particular solution of the differential equation* $x\frac{dy}{dx} - y + x\,\text{cosec}\,\frac{y}{x} = 0$, *given that* $y = 0$ *when* $x = 1$. **[CBSE 2009, '14C]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{y}{x} - \text{cosec}\,\frac{y}{x}. \qquad \ldots (i)$$

This is of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$. So, it is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = v - \text{cosec}\, v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\text{cosec}\, v$$

$$\Rightarrow \quad -\sin v\, dv = \frac{1}{x}dx$$

$$\Rightarrow \quad \int(-\sin v)\, dv = \int\frac{1}{x}dx \quad \text{[on integrating both sides]}$$

$$\Rightarrow \quad \cos v = \log|x| + C, \text{ where } C \text{ is an arbitrary constant}$$

$$\Rightarrow \quad \cos\frac{y}{x} = \log|x| + C \qquad \ldots \text{(ii)} \qquad \left[\because \ v = \frac{y}{x}\right].$$

Putting $x = 1$ and $y = 0$ in (ii), we get $C = 1$.

Hence, $\cos\frac{y}{x} = 1 + \log|x|$ is the required solution.

EXAMPLE 18 *Find the particular solution of the differential equation* $\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\, dy = 0$, *given that* $y = \frac{\pi}{4}$ *when* $x = 1$.

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{y}{x} - \sin^2\left(\frac{y}{x}\right). \qquad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$. So, it is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = v - \sin^2 v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\sin^2 v$$

$$\Rightarrow \quad -\text{cosec}^2 v\, dv = \frac{1}{x}dx$$

$$\Rightarrow \quad \int(-\text{cosec}^2 v)\, dv = \int\frac{1}{x}dx \quad \text{[on integrating both sides]}$$

$$\Rightarrow \quad \cot v = \log|x| + C, \text{ where } C \text{ is an arbitrary constant}$$

$$\Rightarrow \quad \cot\frac{y}{x} = \log|x| + C \qquad \ldots \text{(ii)} \qquad \left[\because \ v = \frac{y}{x}\right].$$

Putting $x = 1$ and $y = \frac{\pi}{4}$ in (ii), we get $C = 1$.

$\therefore \quad \cot \frac{y}{x} = \log |x| + 1$ is the desired solution.

EXAMPLE 19 *Find the equation of the family of curves for which the slope of tangent at any point (x, y) on it, is* $\frac{x^2 + y^2}{2xy}$.

SOLUTION We know that the slope of the tangent at any point (x, y) of the curve is $\frac{dy}{dx}$.

$$\therefore \quad \frac{dy}{dx} = \frac{x^2 + y^2}{2xy}. \qquad \ldots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by x^2, we get

$$\frac{dy}{dx} = \frac{\left\{1 + \left(\frac{y}{x}\right)^2\right\}}{2\left(\frac{y}{x}\right)} = f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \left\{\frac{(1+v^2)}{2v} - v\right\} = \frac{(1-v^2)}{2v}$$

$$\Rightarrow \quad \frac{2v}{(1-v^2)}dv = \frac{1}{x}dx$$

$$\Rightarrow \quad \int \frac{2v}{(v^2-1)}dv = -\int \frac{1}{x}dx \quad \text{[on integrating both sides]}$$

$$\Rightarrow \quad \log |v^2 - 1| = -\log |x| + \log |C_1|,$$

where C_1 is an arbitrary constant

$$\Rightarrow \quad \log |v^2 - 1| + \log |x| = \log |C_1|$$

$$\Rightarrow \quad \log |(v^2 - 1)x| = \log |C_1|$$

$$\Rightarrow \quad (v^2 - 1)x = \pm C_1$$

$$\Rightarrow \quad \left(\frac{y^2}{x^2} - 1\right)x = \pm C_1$$

$$\Rightarrow \quad (y^2 - x^2) = \pm C_1 x$$

$$\Rightarrow \quad (x^2 - y^2) = Cx, \text{ where } \pm C_1 = C.$$

Hence, $(x^2 - y^2) = Cx$ is the equation of the required family of curves.

EXAMPLE 20 *Show that the differential equation* $2ye^{x/y}dx + (y - 2xe^{x/y})dy = 0$ *is homogeneous. Find the particular solution of this differential equation, given that* $x = 0$ *when* $y = 1$. **[CBSE 2013]**

SOLUTION The given differential equation may be written as

$$\frac{dx}{dy} = \frac{(2xe^{x/y} - y)}{2ye^{x/y}}. \qquad \ldots (i)$$

On dividing the Nr and Dr of RHS of (i) by y, we get

$$\frac{dx}{dy} = \frac{\left\{2\frac{x}{y} \cdot e^{x/y} - 1\right\}}{2e^{x/y}} = f\left(\frac{x}{y}\right). \qquad \ldots (ii)$$

So, the given differential equation is homogeneous.

Putting $x = vy$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$ in (ii), we get

$$v + y\frac{dv}{dy} = \frac{(2ve^v - 1)}{2e^v}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \left\{\frac{(2ve^v - 1)}{2e^v} - v\right\} = \frac{-1}{2e^v}$$

$$\Rightarrow \quad 2e^v dv = \frac{-1}{y}dy$$

$$\Rightarrow \quad 2\int e^v dv = -\int \frac{1}{y}dy \quad \text{[on integrating both sides]}$$

$$\Rightarrow \quad 2e^v = -\log|y| + C$$

$$\Rightarrow \quad 2e^{x/y} + \log|y| = C \qquad \ldots (iii) \quad \left[\because \ v = \frac{x}{y}\right].$$

Putting $y = 1$ and $x = 0$ in (iii), we get $C = 2$.

$\therefore \quad 2e^{x/y} + \log|y| = 2$ is the required solution.

EXAMPLE 21 *Show that the differential equation* $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$ *is homogeneous and solve it.*

SOLUTION The given differential equation may be written as

$$\frac{dx}{dy} = \frac{e^{x/y}\left(\frac{x}{y} - 1\right)}{(1 + e^{x/y})}. \qquad \ldots (i)$$

This is of the form $\frac{dx}{dy} = f\left(\frac{x}{y}\right)$. So, it is homogeneous.

Putting $x = vy$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$ in (i), we get

$$v + y\frac{dv}{dy} = \frac{e^v(v-1)}{(1+e^v)}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \left\{\frac{e^v(v-1)}{(1+e^v)} - v\right\} = \frac{-(v+e^v)}{(1+e^v)}$$

$$\Rightarrow \quad \frac{(1+e^v)}{(v+e^v)}dv = -\frac{1}{y}dy$$

$$\Rightarrow \quad \int \frac{(1+e^v)}{(v+e^v)}dv = -\int \frac{1}{y}dy \quad \text{[on integrating both sides]}$$

$$\Rightarrow \quad \log|v+e^v| = -\log|y| + \log|C_1|,$$

where C_1 is an arbitrary constant

$$\Rightarrow \quad \log|v+e^v| + \log|y| = \log|C_1|$$

$$\Rightarrow \quad \log|(v+e^v)y| = \log|C_1|$$

$$\Rightarrow \quad (v+e^v)y = \pm C_1 = C \text{ (say)}$$

$$\Rightarrow \quad \left(\frac{x}{y} + e^{x/y}\right)y = C$$

$$\Rightarrow \quad x + ye^{x/y} = C, \text{ which is the required solution.}$$

EXAMPLE 22 *Show that the differential equation* $y\,dx + x\log\left(\frac{y}{x}\right)dy - 2x\,dy = 0$ *is homogeneous and solve it.*

SOLUTION The given differential equation may be written as

$$y\frac{dx}{dy} = 2x - x\log\left(\frac{y}{x}\right)$$

$$= 2x - x\cdot\log\left\{\frac{1}{(x/y)}\right\}$$

$$= 2x - x\cdot\left[\log 1 - \log\frac{x}{y}\right]$$

$$= 2x + x\log\frac{x}{y}.$$

$$\therefore \quad \frac{dx}{dy} = 2\left(\frac{x}{y}\right) + \left(\frac{x}{y}\right)\log\frac{x}{y} = f\left(\frac{x}{y}\right). \qquad \ldots \text{(i)}$$

So, the given differential equation is homogeneous.

Putting $x = vy$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$ in (i), we get

$$v + y\frac{dv}{dy} = 2v + v(\log v)$$

$$\Rightarrow \quad y\frac{dv}{dy} = v(1+\log v)$$

$$\Rightarrow \int \frac{1}{v(1+\log v)}dv = \int \frac{1}{y}dy \Rightarrow \int \frac{1}{t}dt = \int \frac{1}{y}dy, \text{ where } (1+\log v) = t$$

$$\Rightarrow \log|t| = \log|y| + \log|C_1| \Rightarrow \log|1+\log v| - \log|y| = \log|C_1|$$

$$\Rightarrow \log\left|\frac{1+\log v}{y}\right| = \log|C_1| \Rightarrow \frac{1+\log v}{y} = \pm C_1 = C$$

$\therefore \quad 1+\log\frac{x}{y} = Cy$ is the required solution.

EXERCISE 20

In each of the following differential equations show that it is homogeneous and solve it.

1. $x\,dy = (x+y)\,dx$
2. $(x^2-y^2)\,dx + 2xy\,dy = 0$
3. $x^2dy + y(x+y)\,dx = 0$ **[CBSE 2010]**
4. $(x-y)dy - (x+y)dx = 0$
5. $(x+y)\,dy + (y-2x)\,dx = 0$
6. $(x^2+3xy+y^2)\,dx - x^2dy = 0$ **[CBSE 2006]**
7. $2xy\,dx + (x^2+2y^2)dy = 0$ **[CBSE 2008]**
8. $\frac{dy}{dx} + \frac{x-2y}{2x-y} = 0$
9. $\frac{dy}{dx} + \frac{x^2-y^2}{3xy} = 0$
10. $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$
11. $\frac{dy}{dx} = \frac{2xy}{(x^2-y^2)}$
12. $x^2\frac{dy}{dx} = 2xy + y^2$
13. $x^2\frac{dy}{dx} = x^2 + xy + y^2$
14. $y^2 + (x^2-xy)\frac{dy}{dx} = 0$
15. $x\frac{dy}{dx} - y = 2\sqrt{y^2-x^2}$
16. $y^2dx + (x^2+xy+y^2)dy = 0$
17. $(x-y)\frac{dy}{dx} = x+3y$
18. $(x^3+3xy^2)dx + (y^3+3x^2y)dy = 0$
19. $(x-\sqrt{xy})\,dy = y\,dx$
20. $x^2\frac{dy}{dx} + y^2 = xy$
21. $x\frac{dy}{dx} = y(\log y - \log x + 1)$ **[CBSE 2007]**
22. $x\frac{dy}{dx} - y + x\sin\frac{y}{x} = 0$ **[CBSE 2012]**
23. $x\frac{dy}{dx} = y - x\cos^2\left(\frac{y}{x}\right)$
24. $\left(x\cos\frac{y}{x}\right)\frac{dy}{dx} = \left(y\cos\frac{y}{x}\right) + x$ **[CBSE 2012C]**
25. Find the particular solution of the differential equation

    $2xy + y^2 - 2x^2\frac{dy}{dx} = 0$, it being given that $y = 2$ when $x = 1$.

26. Find the particular solution of the differential equation $\left\{x \sin^2 \frac{y}{x} - y\right\} dx + x\, dy = 0$, it being given that $y = \frac{\pi}{4}$ when $x = 1$.

[CBSE 2011, '14C]

27. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{y(2y - x)}{x(2y + x)}$, given that $y = 1$ when $x = 1$.

28. Find the particular solution of the differential equation $(xe^{y/x} + y)\, dx = x\, dy$, given that $y(1) = 0$.

29. Find the particular solution of the differential equation $xe^{y/x} - y + x\frac{dy}{dx} = 0$, given that $y(e) = 0$.

30. The slope of the tangent to a curve at any point (x, y) on it is given by $\frac{y}{x} - \left(\cot \frac{y}{x}\right)\left(\cos \frac{y}{x}\right)$, where $x > 0$ and $y > 0$. If the curve passes through the point $\left(1, \frac{\pi}{4}\right)$, find the equation of the curve.

ANSWERS (EXERCISE 20)

1. $y = x \log |x| + Cx$ **2.** $x^2 + y^2 = Cx$ **3.** $x^2y = C^2(y + 2x)$

4. $\tan^{-1} \frac{y}{x} = \frac{1}{2} \log (x^2 + y^2) + C$ **5.** $y^2 + 2xy - 2x^2 = C$

6. $\log |x| + \frac{x}{(y + x)} = C$ **7.** $3x^2y + 2y^3 = C$ **8.** $(y - x) = C(y + x)^3$

9. $(x^2 + 2y^2)^3 = Cx^2$ **10.** $(x^2 - y^2) = Cx$ **11.** $y = C(y^2 + x^2)$

12. $y = Cx(x + y)$ **13.** $\tan^{-1} \frac{y}{x} = \log |x| + C$ **14.** $y = x\{C + \log |y|\}$

15. $y + \sqrt{y^2 - x^2} = C|x|^3$ **16.** $\log \left|\frac{y}{y + x}\right| + \log |x| + \frac{x}{(y + x)} = C$

17. $\log |x + y| + \frac{2x}{(x + y)} = C$ **18.** $y^4 + 6x^2y^2 + x^4 = C$

19. $2\sqrt{\frac{x}{y}} + \log |y| = C$ **20.** $Cx = e^{x/y}$ **21.** $y = xe^{Cx}$

22. $x \tan \left(\frac{y}{2x}\right) = C$ **23.** $\tan \frac{y}{x} + \log |x| = C$

24. $\sin \left(\frac{y}{x}\right) = \log |x| + C$ **25.** $y = \frac{2x}{(1 - \log |x|)}$ **26.** $\cot \frac{y}{x} = \log |x| + 1$

27. $\frac{y}{x} + \frac{1}{2} \log |xy| = 1$ **28.** $\log |x| + e^{-y/x} = 1$ **29.** $y = -x \log (\log |x|)$

30. $\sec\frac{y}{x}+\log|x|=\sqrt{2}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 20)

2. $\frac{dy}{dx}=\frac{y^2-x^2}{2xy}$, which is homogeneous.

$\int\frac{2v}{(1+v^2)}dv=\int\frac{-1}{x}dx \Rightarrow \log|1+v^2|+\log|x|=\log|C|.$

3. $\int\frac{dv}{v(v+2)}=-\int\frac{1}{x}dx$

$\Rightarrow \frac{1}{2}\int\left\{\frac{1}{v}-\frac{1}{(v+2)}\right\}dv=-\int\frac{1}{x}dx$ [by partial fractions]

4. $\int\left(\frac{1-v}{1+v^2}\right)dv=\int\frac{1}{x}dx \Rightarrow \int\frac{dv}{(1+v^2)}-\frac{1}{2}\int\frac{2v}{(1+v^2)}dv=\int\frac{1}{x}dx$

$\therefore \tan^{-1}v-\frac{1}{2}\log(1+v^2)=\log|x|+C \Rightarrow \tan^{-1}\frac{y}{x}=\frac{1}{2}\log(x^2+y^2)+C.$

5. $\int\frac{(v+1)}{(v^2+2v-2)}dv=-\int\frac{dx}{x} \Rightarrow \frac{1}{2}\int\frac{(2v+2)}{(v^2+2v-2)}dv=-\int\frac{dx}{x}$

$\therefore \frac{1}{2}\log|v^2+2v-2|+\log|x|=\log|C_1|.$

6. $x\frac{dv}{dx}=(1+2v+v^2)=(v+1)^2 \Rightarrow \int\frac{dv}{(v+1)^2}=\int\frac{1}{x}dx \Rightarrow \log|x|-\frac{1}{(v+1)}=C.$

7. $x\frac{dv}{dx}=\frac{-(3v+2v^3)}{(1+2v^2)} \Rightarrow \int\frac{(1+2v^2)}{(3v+2v^3)}dv+\int\frac{dx}{x}=\log|C_1|.$

$\therefore \frac{1}{3}\int\frac{(3+6v^2)}{(3v+2v^3)}dv+\log|x|=\log|C_1|$

$\Rightarrow \frac{1}{3}\log|3v+2v^3|+\log|x|=\log|C_1|$

$\Rightarrow \log|3v+2v^3|+3\log|x|=3\log|C_1|$

$\Rightarrow \log|3v+2v^3|+\log|x^3|=\log|C_1^3|.$

8. $\int\frac{(2-v)}{(v^2-1)}dv=\int\frac{1}{x}dx.$

$\frac{(2-v)}{(v^2-1)}=\left\{\frac{1}{2(v-1)}-\frac{3}{2(v+1)}\right\}$ [by partial fractions].

9. $x\frac{dv}{dx}=\frac{-(1+2v^2)}{3v} \Rightarrow \int\frac{3v}{(1+2v^2)}dv=-\int\frac{1}{x}dx$

$\Rightarrow \frac{3}{4}\int\frac{4v}{(1+2v^2)}dv=-\int\frac{1}{x}dx \Rightarrow \frac{3}{4}\log|1+2v^2|+\log|x|=\log|C_1|.$

10. $x\frac{dv}{dx}=\frac{(1-v^2)}{2v} \Rightarrow \int \frac{-2v}{(1-v^2)}dv=-\int\frac{1}{x}dx$

$\Rightarrow \log|1-v^2|+\log|x|=\log|C_1|.$

11. $x\frac{dv}{dx}=\frac{(v+v^3)}{(1-v^2)} \Rightarrow \int \frac{(1-v^2)}{v(1+v^2)}dv=\int\frac{1}{x}dx$

$\Rightarrow \int\left\{\frac{1}{v}-\frac{2v}{(1+v^2)}\right\}dv=\int\frac{1}{x}dx$ [by partial fractions].

12. $\int\frac{1}{v(1+v)}dv=\int\frac{1}{x}dx \Rightarrow \int\left\{\frac{1}{v}-\frac{1}{(1+v)}\right\}dv=\int\frac{1}{x}dx$ [by partial fractions].

13. $\int\frac{1}{(1+v^2)}dv=\int\frac{1}{x}dx \Rightarrow \tan^{-1}v=\log|x|+C.$

14. $\int\frac{(v-1)}{v(v+1)}dv=\int\frac{1}{x}dx \Rightarrow \int\left\{\frac{2}{v+1}-\frac{1}{v}\right\}dv=\int\frac{1}{x}dx.$

15. $\int\frac{1}{\sqrt{v^2-1}}dv=\int\frac{2}{x}dx \Rightarrow \log|v+\sqrt{v^2-1}|=2\log|x|+\log|C_1|.$

16. $\int\frac{1}{v(v+1)^2}dv=-\int\frac{1}{x}dx.$

Let $\frac{1}{v(v+1)^2}=\frac{A}{v}+\frac{B}{(v+1)}+\frac{C}{(v+1)^2}.$

This gives $A=1, B=-1$ and $C=-1$.

17. $x\frac{dv}{dx}=\frac{(1+v)^2}{(1-v)} \Rightarrow \int\frac{(v-1)}{(v+1)^2}dv=-\int\frac{dx}{x}$

$\Rightarrow \int\left\{\frac{(v+1)-2}{(v+1)^2}\right\}dv=-\int\frac{dx}{x} \Rightarrow \int\frac{1}{(v+1)}dv-2\int\frac{1}{(v+1)^2}dv+\int\frac{dx}{x}=C.$

18. $\int\frac{(v^3+3v)}{(v^4+6v^2+1)}dv=-\int\frac{1}{x}dx$

$\Rightarrow \frac{1}{4}\int\frac{(4v^3+12v)}{(v^4+6v^2+1)}dv=-\int\frac{1}{x}dx$

$\Rightarrow \frac{1}{4}\log|v^4+6v^2+1|+\log|x|=\log|C_1|$

$\Rightarrow (v^4+6v^2+1)x^4=C_1^4=C$ (say) $\Rightarrow y^4+6x^2y^2+x^4=C.$

19. $\int\frac{1-\sqrt{v}}{v^{3/2}}dv=\int\frac{1}{x}dx \Rightarrow \int\left\{\frac{1}{v^{3/2}}-\frac{\sqrt{v}}{v^{3/2}}\right\}dv=\int\frac{1}{x}dx$

$\Rightarrow \int v^{-3/2}dv-\int\frac{1}{v}dv=\log|x|-C \Rightarrow \frac{-2}{\sqrt{v}}-\log|v|=\log|x|-C$

$\Rightarrow 2\sqrt{\frac{x}{y}}+\log|y|=C.$

20. $x\dfrac{dv}{dx} = -v^2 \Rightarrow \int -\dfrac{1}{v^2}dv = \int \dfrac{1}{x}dx \Rightarrow \dfrac{1}{v} = \log x + \log C.$

$\therefore \quad \log Cx = \dfrac{x}{y} \Rightarrow Cx = e^{x/y}.$

21. $\dfrac{dy}{dx} = \dfrac{y}{x}\left\{\log \dfrac{y}{x} + 1\right\} = f\left(\dfrac{y}{x}\right).$

Putting $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$, we get

$$\int \frac{1}{v\log v}dv = \int \frac{1}{x}dx \Rightarrow \log(\log v) = \log x + \log C$$

$\therefore \quad \log v = Cx \Rightarrow v = e^{Cx} \Rightarrow y = xe^{Cx}.$

22. $\dfrac{dy}{dx} = \dfrac{y}{x} - \sin\dfrac{y}{x} = f\left(\dfrac{y}{x}\right).$

$$\int \operatorname{cosec} v\,dv = \int \frac{-1}{x}dx \Rightarrow \log\tan\frac{v}{2} + \log x = \log C.$$

25. $\dfrac{dy}{dx} = \dfrac{2xy + y^2}{2x^2} = \left\{\left(\dfrac{y}{x}\right) + \dfrac{1}{2}\left(\dfrac{y}{x}\right)^2\right\} = f\left(\dfrac{y}{x}\right).$

$$2\int \frac{1}{v^2}dv = \int \frac{1}{x}dx \Rightarrow \frac{-2}{v} = \log|x| + C \Rightarrow \frac{-2x}{y} = \log|x| + C.$$

Putting $x = 1$ and $y = 2$, we get $C = -1$.

30. $\dfrac{dy}{dx} = \left\{\dfrac{y}{x} - \cot\dfrac{y}{x}\cos\dfrac{y}{x}\right\} = f\left(\dfrac{y}{x}\right)$ which is homogeneous.

Putting $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$, we get

$$\int \sec v\tan v\,dv = -\int \frac{dx}{x} \Rightarrow \sec v = -\log|x| + C$$

$\therefore \quad \sec\dfrac{y}{x} + \log|x| = C.$

Putting $x = 1$ and $y = \dfrac{\pi}{4}$, we get $C = \sqrt{2}$.

21. LINEAR DIFFERENTIAL EQUATIONS

Linear Differential Equations

The most general form of linear differential equations is $\frac{dy}{dx} + Py = Q$, where P is a constant and Q is a constant or a function of x. The other common form of linear differential equations is $\frac{dx}{dy} + Px = Q$, where P is a constant and Q is a constant or a function of y.

Solution of $\frac{dy}{dx} + Py = Q$

First, we find $e^{\int P\,dx}$, which is known as the *integrating factor*, i.e., IF.

Now, $\frac{dy}{dx} + Py = Q \;\Rightarrow\; e^{\int P\,dx}\frac{dy}{dx} + Py\,e^{\int P\,dx} = Q \cdot e^{\int P\,dx}$

$\Rightarrow\; e^{\int P\,dx}dy + Py \cdot e^{\int P\,dx}dx = Q \cdot e^{\int P\,dx}dx \;\Rightarrow\; d\left(y \cdot e^{\int P\,dx}\right) = Q \cdot e^{\int P\,dx}dx$ … (i)

$\Rightarrow\; y \cdot e^{\int P\,dx} = \int Q \cdot e^{\int P\,dx}dx + C$, which is the required solution

[integrating both sides of (i)].

Working Rule for Solving $\frac{dy}{dx} + Py = Q$

(i) Find $\text{IF} = e^{\int P\,dx}$.

(ii) The solution is given by

$$y \times \text{IF} = \int \{Q \times \text{IF}\}dx + C.$$

SOLVED EXAMPLES

EXAMPLE 1 *Solve the differential equation*

$$x\frac{dy}{dx} - y = 2x^3,\ x > 0.$$ **[CBSE 2004, '07, '11]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} + \left(\frac{-1}{x}\right)y = 2x^2. \quad \text{… (i)}$$

This is of the form, $\frac{dy}{dx} + Py = Q$, where $P = \frac{-1}{x}$ and $Q = 2x^2$.

Thus, the given differential equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log\left(\frac{1}{x}\right)} = \frac{1}{x}.$$

So, the required solution is given by

$$y \times \text{IF} = \int \{Q \times \text{IF}\}\,dx + C, \text{ where } C \text{ is an arbitrary constant,}$$

$$\text{i.e., } y \times \frac{1}{x} = \int \left(2x^2 \times \frac{1}{x}\right) dx + C = 2\int x\,dx + C = x^2 + C$$

$$\Rightarrow \quad y = x^3 + Cx.$$

Hence, the required solution is $y = x^3 + Cx$.

EXAMPLE 2 *Solve the differential equation*

$$(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{2}{(x^2 - 1)}.$$ **[CBSE 2010, '14]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} + \left\{\frac{2x}{(x^2 - 1)}\right\} y = \frac{2}{(x^2 - 1)^2}. \quad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$,

where $P = \frac{2x}{(x^2 - 1)}$ and $Q = \frac{2}{(x^2 - 1)^2}$.

Thus, the given differential equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{\int \frac{2x}{(x^2 - 1)}dx} = e^{\log(x^2 - 1)} = (x^2 - 1).$$

So, the required solution is given by

$$y \times \text{IF} = \int \{Q \times \text{IF}\}\,dx + C$$

$$\text{i.e., } y \times (x^2 - 1) = \int \left\{\frac{2}{(x^2 - 1)^2} \times (x^2 - 1)\right\} dx + C$$

$$= 2\int \frac{dx}{(x^2 - 1)} + C$$

$$= 2\int \frac{1}{2}\left\{\frac{1}{(x - 1)} - \frac{1}{(x + 1)}\right\} dx + C \quad \text{[by partial fraction]}$$

$$= \log\left|\frac{x - 1}{x + 1}\right| + C.$$

Hence, $y(x^2 - 1) = \log\left|\frac{x - 1}{x + 1}\right| + C$ is the required solution.

EXAMPLE 3 *Solve* $(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$. **[CBSE 2014]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx}+\frac{1}{(1+x^2)}\cdot y=\frac{e^{\tan^{-1}x}}{(1+x^2)}.$$

This is of the form $\frac{dy}{dx}+Py=Q$, where $P=\frac{1}{(1+x^2)}$ and $Q=\frac{e^{\tan^{-1}x}}{(1+x^2)}$.

Thus, the given equation is linear.

$$\text{IF}=e^{\int P\,dx}=e^{\int\frac{1}{(1+x^2)}dx}=e^{\tan^{-1}x}.$$

So, the required solution is given by

$$y\times\text{IF}=\int|(Q\times(\text{IF})|\,dx+C,$$

$$\text{i.e., } y\times e^{\tan^{-1}x}=\int\left\{\frac{e^{\tan^{-1}x}}{(1+x^2)}\times e^{\tan^{-1}x}\right\}dx+C$$

$$=\int\frac{e^{2\tan^{-1}x}}{(1+x^2)}dx+C$$

$$=\int e^{2t}\,dt+C,\text{ where }\tan^{-1}x=t$$

$$=\frac{1}{2}e^{2t}+C=\frac{1}{2}e^{2\tan^{-1}x}+C.$$

Hence, $y=\frac{1}{2}e^{\tan^{-1}x}+Ce^{-\tan^{-1}x}$ is the required solution.

EXAMPLE 4 *Solve* $(x\log x)\frac{dy}{dx}+y=\frac{2}{x}\log x.$ **[CBSE 2009, '10, '14]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx}+\frac{1}{(x\log x)}\cdot y=\frac{2}{x^2}.$$

This is of the form $\frac{dy}{dx}+Py=Q$, where $P=\frac{1}{(x\log x)}$ and $Q=\frac{2}{x^2}$.

Thus, the given differential equation is linear.

$$\text{IF}=e^{\int P\,dx}=e^{\int\frac{1}{x\log x}dx}=e^{\int\frac{1}{t}dt},\text{ where }\log x=t$$

$$=e^{\log t}=t=\log x.$$

So, the solution of the given differential equation is

$$y\times\text{IF}=\int\{Q\times(\text{IF})\}\,dx+C,$$

$$\text{i.e., } y(\log x)=\int\left(\frac{2}{x^2}\log x\right)dx+C$$

$$=2\int(\log x)\cdot\frac{1}{x^2}dx+C$$

$$=2\left[(\log x)\left(-\frac{1}{x}\right)-\int\frac{1}{x}\cdot\left(-\frac{1}{x}\right)dx\right]+C$$

[integrating by parts]

$$= \frac{-2\log x}{x} - \frac{2}{x} + C.$$

Hence, $y(\log x) = \frac{-2}{x}(\log x + 1) + C$ is the required solution.

EXAMPLE 5 *Solve* $(x^2+1)\frac{dy}{dx} + 2xy = \sqrt{x^2+4}$. **[CBSE 2008, '10]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} + \left(\frac{2x}{x^2+1}\right)y = \frac{\sqrt{x^2+4}}{(x^2+1)}. \quad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$,

where $P = \frac{2x}{(x^2+1)}$ and $Q = \frac{\sqrt{x^2+4}}{(x^2+1)}$.

Thus, the given differential equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{\int \frac{2x}{(x^2+1)}dx} = e^{\log(x^2+1)} = (x^2+1).$$

So, the required solution is given by

$$y \times \text{IF} = \int \{Q \times \text{IF}\}\,dx + C,$$

$$\text{i.e., } y(x^2+1) = \int \frac{\sqrt{x^2+4}}{(x^2+1)} \times (x^2+1)\,dx$$

$$\Rightarrow \quad y(x^2+1) = \int \sqrt{x^2+4}\,dx$$

$$= \frac{1}{2}x\sqrt{x^2+4} + \frac{1}{2} \times 2^2 \times \log|x + \sqrt{x^2+4}| + C$$

$$= \frac{1}{2}x\sqrt{x^2+4} + 2\log|x + \sqrt{x^2+4}| + C.$$

Hence, $y(x^2+1) = \frac{1}{2}x\sqrt{x^2+4} + 2\log|x + \sqrt{x^2+4}| + C$ is the required solution.

EXAMPLE 6 *Solve* $(1+x^2)\frac{dy}{dx} + y = \tan^{-1}x$. **[CBSE 2008C, '09]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} + \frac{1}{(1+x^2)} \cdot y = \frac{\tan^{-1}x}{(1+x^2)}. \quad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{1}{(1+x^2)}$ and $Q = \frac{\tan^{-1}x}{(1+x^2)}$.

Thus, the given equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1}x}.$$

$\therefore$ the required solution is

$$y \times \text{IF} = \int \{Q \times \text{IF}\}\, dx + C,$$

i.e., $y \times e^{\tan^{-1}x} = \int \left\{ \dfrac{\tan^{-1}x}{(1+x^2)} \cdot e^{\tan^{-1}x} \right\} dx + C$

$= \int (t\, e^t)\, dt + C$, where $\tan^{-1} x = t$

$= t\, e^t - \int 1 \cdot e^t\, dt + C$ [integrating by parts]

$= t\, e^t - e^t + C = e^t(t-1) + C$

$= e^{\tan^{-1}x}(\tan^{-1} x - 1) + C$

$\Rightarrow \quad y = (\tan^{-1}x - 1) + Ce^{-\tan^{-1}x}$

Hence, $y = (\tan^{-1}x - 1) + Ce^{-\tan^{-1}x}$ is the required solution.

EXAMPLE 7 *Find the general solution of the differential equation*

$$\frac{dy}{dx} - y = \cos x. \qquad \textbf{[CBSE 2012C]}$$

SOLUTION The given differential equation is

$$\frac{dy}{dx} - y = \cos x. \qquad \ldots \text{(i)}$$

This is of the form $\dfrac{dy}{dx} + Py = Q$, where $P = -1$ and $Q = \cos x$.

Thus, the given differential equation is linear.

$\text{IF} = e^{\int P\, dx} = e^{\int -dx} = e^{-x}$.

So, the required solution is

$$y \times \text{IF} = \int \{Q \times \text{IF}\}\, dx + C,$$

i.e., $y \times e^{-x} = \int (\cos x) e^{-x}\, dx + C.$ $\ldots$ (ii)

Let $I = \int (\cos x)\, e^{-x}\, dx$ (I, II)

$= (\cos x)(-e^{-x}) - \int (-\sin x)(-e^{-x})\, dx$ [integrating by parts]

$= -(\cos x)e^{-x} - \int (\sin x)(e^{-x})\, dx$ (I, II)

$= -(\cos x)e^{-x} - \{(\sin x)(-e^{-x})\} - \int (\cos x)(-e^{-x})\, dx$

$= -(\cos x)e^{-x} + (\sin x)(e^{-x}) - I.$

$\therefore \quad 2I = (\sin x - \cos x)e^{-x} \Rightarrow I = \dfrac{1}{2}(\sin x - \cos x)e^{-x}.$

Putting this value in (ii), we get

$$y \times e^{-x} = \frac{1}{2}(\sin x - \cos x)e^{-x} + C \Rightarrow y = \frac{1}{2}(\sin x - \cos x) + Ce^{x}.$$

Hence, $y = \dfrac{1}{2}(\sin x - \cos x) + Ce^{x}$ is the required solution.

EXAMPLE 8 *Solve the differential equation*

$$x\frac{dy}{dx} + y - x + xy\cot x = 0,\ x \neq 0. \qquad \textbf{[CBSE 2011, '12]}$$

SOLUTION The given differential equation may be written as

$$x\frac{dy}{dx} + (1 + x\cot x)\,y = x$$

$$\Rightarrow \quad \frac{dy}{dx} + \left(\frac{1}{x} + \cot x\right)y = 1. \qquad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \left(\frac{1}{x} + \cot x\right)$ and $Q = 1$.

Thus, the given differential equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{\int\left(\frac{1}{x} + \cot x\right)dx} = e^{\log x + \log \sin x}$$

$$= e^{\log (x \sin x)} = x \sin x.$$

So, the solution of the given differential equation is given by

$$y \times (\text{IF}) = \int (Q \times \text{IF})\,dx + C,$$

$$\text{i.e., } y(x \sin x) = \int 1 \times (x \sin x)\,dx + C$$

$$= \int \underset{\text{I}}{x}\ \underset{\text{II}}{\sin x}\,dx + C$$

$$= x(-\cos x) - \int 1 \cdot (-\cos x)\,dx + C$$

$$= -x\cos x + \int \cos x\,dx + C$$

$$= -x\cos x + \sin x + C.$$

$$\therefore \quad y = \frac{1}{x} - \cot x + \frac{C}{x \sin x} \text{ is the required solution.}$$

EXAMPLE 9 *Solve the differential equation*

$$\left\{\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right\}\frac{dx}{dy} = 1\ (x \neq 0). \qquad \textbf{[CBSE 2012]}$$

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = -\frac{1}{\sqrt{x}} \cdot y + \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{\sqrt{x}} \cdot y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \cdot \qquad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{1}{\sqrt{x}}$ and $Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \cdot$

$$\text{IF} = e^{\int P\,dx} = e^{\int \frac{1}{\sqrt{x}}dx} = e^{2\sqrt{x}}.$$

So, the solution of the given differential equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\, dx + C,$$

i.e., $y \times e^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \times e^{2\sqrt{x}}\, dx + C$

$$= \int \frac{1}{\sqrt{x}}\, dx + C = 2\sqrt{x} + C.$$

Hence, $ye^{2\sqrt{x}} = 2\sqrt{x} + C$ is the required solution.

$\therefore \quad y = \frac{1}{x} - \cot x + \frac{C}{x \sin x}$ is the required solution.

EXAMPLE 10 *Solve* $x\frac{dy}{dx} + 2y = x \cos x.$

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} + \frac{2}{x} \cdot y = \cos x. \qquad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{2}{x}$ and $Q = \cos x$.

Thus, the given equation is linear.

$$\text{IF} = e^{\int P\, dx} = e^{\int \frac{2}{x} dx} = e^{2 \log x} = e^{\log (x^2)} = x^2.$$

So, the required solution is

$$y \times \text{IF} = \int \{Q \times (\text{IF})\}\, dx + C,$$

i.e., $yx^2 = \int x^2 \cos x\, dx + C$

$= x^2 \sin x - \int 2x \sin x\, dx + C$ [integrating by parts]

$= x^2 \sin x - 2 \cdot [x(-\cos x) - \int 1 \cdot (-\cos x)\, dx] + C$

[integrating by parts]

$= x^2 \sin x + 2x \cos x - 2 \sin x + C.$

Hence, $y = \sin x + \frac{2}{x} \cos x - \frac{2}{x^2} \sin x + \frac{C}{x^2}$ is the required solution.

EXAMPLE 11 *Solve* $\frac{dy}{dx} + (\sec x) y = \tan x.$ **[CBSE 2006, '08C, '12]**

SOLUTION The given equation is of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \sec x \text{ and } Q = \tan x.$$

Thus, the given equation is linear.

$$\text{IF} = e^{\int P\, dx} = e^{\int \sec x\, dx} = e^{\log (\sec x + \tan x)} = (\sec x + \tan x).$$

So, the required solution is

$$y \times \text{IF} = \int \{Q \times (\text{IF})\}\, dx + C,$$

i.e., $y(\sec x + \tan x) = \int \tan x(\sec x + \tan x)\,dx + C$

$= \int \sec x \tan x\,dx + \int \tan^2 x\,dx + C$

$= \sec x + \int (\sec^2 x - 1)\,dx + C$

$= \sec x + \tan x - x + C.$

Hence, $y(\sec x + \tan x) = \sec x + \tan x - x + C$ is the required solution.

EXAMPLE 12 *Find the general solution of the differential equation* $\frac{dy}{dx} - 2y = \cos 3x.$

SOLUTION The given differential equation is of the form $\frac{dy}{dx} + Py = Q$, where $P = -2$ and $Q = \cos 3x$.

Thus, the given differential equation is linear.

$\text{IF} = e^{\int P\,dx} = e^{\int -2dx} = e^{-2x}.$

So, the required solution is

$y \times \text{IF} = \int \{Q \times \text{IF}\}\,dx + C,$

i.e., $y \times e^{-2x} = \int e^{-2x} \cos 3x\,dx + C$

$= e^{-2x}\left[\frac{-2\cos 3x + 3\sin 3x}{\{(-2)^2 + 3^2\}}\right] + C$

$\left[\because \int e^{ax} \cos bx\,dx = e^{ax}\left\{\frac{a\cos bx + b\sin bx}{(a^2 + b^2)}\right\}\right].$

$\therefore \quad y = \frac{(3\sin 3x - 2\cos 3x)}{13} + Ce^{2x}$, which is the required solution.

EXAMPLE 13 *Solve the differential equation*

$$(\cos^2 x)\frac{dy}{dx} + y = \tan x \left(0 \le x < \frac{\pi}{2}\right).$$ **[CBSE 2008, '11]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} + (\sec^2 x)y = (\sec^2 x)\tan x. \quad \ldots (i)$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \sec^2 x$ and $Q = \sec^2 x \tan x$.

Thus, the given differential equation is linear.

$\text{IF} = e^{\int P\,dx} = e^{\int \sec^2 x\,dx} = e^{\tan x}.$

So, its solution is given by

$y \times \text{IF} = \int (Q \times \text{IF})\,dx + C,$

$$\text{i.e., } y \times e^{\tan x} = \int e^{\tan x}(\sec^2 x)\tan x\,dx + C$$

$$= \int \underset{\text{I}}{t}\,\underset{\text{II}}{e^t}\,dt, \text{ where } \tan x = t \text{ and } \sec^2 x\,dx = dt$$

$$= te^t - \int 1 \cdot e^t dt + C$$

$$= te^t - e^t + C = e^t(t-1) + C$$

$$= e^{\tan x}(\tan x - 1) + C.$$

Hence, $y = (\tan x - 1) + Ce^{-\tan x}$ is the required solution.

EXAMPLE 14 *Solve the differential equation*

$$\frac{dy}{dx} + y \tan x = 2x + x^2 \tan x.$$

SOLUTION The given differential equation is of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \tan x \text{ and } Q = 2x + x^2 \tan x.$$

Thus, the given differential equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{\int \tan x\,dx} = e^{\log \sec x} = \sec x.$$

So, its solution is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\,dx + C,$$

$$\text{i.e., } y \times \sec x = \int (2x + x^2 \tan x)\sec x\,dx + C$$

$$= \int 2x \sec x\,dx + \int \underset{\text{I}}{x^2}\,\underset{\text{II}}{\sec x \tan x}\,dx + C$$

$$= \int 2x \sec x\,dx + \{x^2 \sec x - \int 2x \sec x\,dx\} + C$$

$$= x^2 \sec x + C.$$

$\therefore$ $y = x^2 + C\cos x$ is the required solution.

EXAMPLE 15 *Solve the differential equation* $x\frac{dy}{dx} + y = x\cos x + \sin x$, *given* $y\left(\frac{\pi}{2}\right) = 1$.

[CBSE 2014C]

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} + \frac{1}{x} \cdot y = \cos x + \frac{\sin x}{x} \cdot \quad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{1}{x}$ and

$Q = \cos x + \frac{\sin x}{x}$.

Thus, the given differential equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x.$$

So, its solution is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\,dx + C,$$

$$\text{i.e., } y \times x = \int \left(\cos x + \frac{\sin x}{x} \right) x \, dx + C$$

$$= \int \underset{\text{I}}{x} \underset{\text{II}}{\cos x} \, dx + \int \sin x \, dx + C$$

$$= x \sin x - \int \sin x \, dx + \int \sin x \, dx + C$$

$$= x \sin x + C.$$

$$\therefore \quad y = \sin x + \frac{C}{x}. \qquad \ldots \text{(ii)}$$

It is being given that when $x = \frac{\pi}{2}$, then $y = 1$.

Putting $x = \frac{\pi}{2}$ and $y = 1$ in (ii), we get $C = 0$.

Hence, $y = \sin x$ is the required solution.

EXAMPLE 16 *Find the particular solution of the differential equation* $\cos x \frac{dy}{dx} + y = \sin x$, *given that* $y = 2$ *when* $x = 0$.

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} + (\sec x) y = \tan x. \qquad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \sec x$ and $Q = \tan x$.

Thus, the given differential equation is linear.

$$\text{IF} = e^{\int P \, dx} = e^{\int \sec x \, dx} = e^{\log (\sec x + \tan x)} = (\sec x + \tan x).$$

So, its solution is given by

$$y \times \text{IF} = \int (Q \times \text{IF}) \, dx + C,$$

$$\text{i.e., } y(\sec x + \tan x) = \int \tan x (\sec x + \tan x) \, dx + C$$

$$= \int \sec x \tan x \, dx + \int \tan^2 x \, dx + C$$

$$= \sec x + \int (\sec^2 x - 1) \, dx + C$$

$$= \sec x + \int \sec^2 x \, dx - \int dx + C$$

$$= \sec x + \tan x - x + C.$$

Thus, $y(\sec x + \tan x) = \sec x + \tan x - x + C$. $\qquad \ldots \text{(ii)}$

It is given that when $x = 0$, then $y = 2$.

$\therefore$ putting $x = 0$ and $y = 2$ in (ii), we get $C = 1$.

Hence, $y(\sec x + \tan x) = \sec x + \tan x - x + 1$ is the required solution.

EXAMPLE 17 *Find the particular solution of the differential equation* $x \frac{dy}{dx} - y = (x + 1) e^{-x}$, *given that* $y = 0$ *when* $x = 1$.

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} - \frac{1}{x} \cdot y = \frac{(x+1)e^{-x}}{x}. \qquad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{-1}{x}$ and $Q = \frac{(x+1)e^{-x}}{x}$.

Thus, the given differential equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = e^{\log(1/x)} = \frac{1}{x}.$$

So, its solution is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\,dx + C,$$

$$\text{i.e., } y \times \frac{1}{x} = \int \frac{(x+1)e^{-x}}{x^2}dx + C$$

$$\Rightarrow \quad y \times \frac{1}{x} = \int \underset{\text{I}}{\frac{1}{x}} \underset{\text{II}}{e^{-x}}\,dx + \int \frac{1}{x^2}e^{-x}dx + C$$

$$\Rightarrow \quad y \times \frac{1}{x} = \frac{1}{x} \cdot (-e^{-x}) + \int \frac{1}{x^2}(-e^{-x})\,dx + \int \frac{1}{x^2}e^{-x}dx + C$$

$$\Rightarrow \quad \frac{y}{x} = \frac{-e^{-x}}{x} + C$$

$$\Rightarrow \quad y = -e^{-x} + Cx. \qquad \ldots \text{(ii)}$$

It is being given that when $x = 1$, then $y = 0$.

Putting $x = 1$ and $y = 0$ in (ii), we get $C = e^{-1}$.

$\therefore \quad y = -e^{-x} + xe^{-1}$ is the required solution.

EXAMPLE 18 *Find the particular solution of the differential equation* $\frac{dy}{dx} - 3y \cot x = \sin 2x$, *it being given that* $y = 2$ *when* $x = \frac{\pi}{2}$.

SOLUTION The given differential equation is

$$\frac{dy}{dx} - 3y \cot x = \sin 2x. \qquad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = -3\cot x$ and $Q = \sin 2x$.

Thus, the given equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{\int -3\cot x\,dx} = e^{\log(\sin x)^{-3}} = (\sin x)^{-3} = \frac{1}{\sin^3 x}.$$

So, the solution of (i) is given by

$$y \times \text{IF} = \int \{Q \times (\text{IF})\}\,dx + C,$$

$$\text{i.e., } y \times \frac{1}{\sin^3 x} = \int \left(\sin 2x \times \frac{1}{\sin^3 x}\right)dx + C$$

$$= 2\int \frac{\cos x}{\sin^2 x}dx + C \qquad [\because \ \sin 2x = 2\sin x \cos x]$$

$$= 2\int \frac{1}{t^2}\,dt + C, \text{ where } \sin x = t$$

$$= \frac{-2}{t} + C = \frac{-2}{\sin x} + C.$$

Thus, $y = -2\sin^2 x + C\sin^3 x.$ … (ii)

It is being given that when $x = \frac{\pi}{2}$, then $y = 2$.

Putting $x = \frac{\pi}{2}$ and $y = 2$ in (ii), we get

$$-2\sin^2\frac{\pi}{2} + C\sin^3\frac{\pi}{2} = 2 \Rightarrow C - 2 = 2 \Rightarrow C = 4.$$

Hence, $y = -2\sin^2 x + 4\sin^3 x$ is the required solution.

EXAMPLE 19 *Find the particular solution of the differential equation* $(1-x^2)\frac{dy}{dx} - xy = x^2$, *given that* $y = 2$ *when* $x = 0$.

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} - \frac{x}{(1-x^2)}\cdot y = \frac{x^2}{(1-x^2)}. \quad \text{… (i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{-x}{(1-x^2)}$ and $Q = \frac{x^2}{(1-x^2)}$.

Thus, the given differential equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{\int \frac{-x}{(1-x^2)}dx} = e^{\frac{1}{2}\int \frac{-2x}{(1-x^2)}dx} = e^{\frac{1}{2}\log(1-x^2)}$$

$$= e^{\log\sqrt{1-x^2}} = \sqrt{1-x^2}$$

So, its solution is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\,dx + C,$$

$$\text{i.e., } y\times\sqrt{1-x^2} = \int \frac{x^2}{(1-x^2)}\times\sqrt{1-x^2}\,dx + C$$

$$= \int \frac{x^2}{\sqrt{1-x^2}}dx + C$$

$$= \int \frac{\{-(1-x^2)+1\}}{\sqrt{1-x^2}}dx + C$$

$$= -\int \sqrt{1-x^2}\,dx + \int \frac{1}{\sqrt{1-x^2}}dx + C$$

$$= -\left\{\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right\} + \sin^{-1}x + C$$

$$= \frac{-x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x + C.$$

$$\therefore \quad y = \frac{-x}{2} + \frac{\sin^{-1}x}{2\sqrt{1-x^2}} + \frac{C}{\sqrt{1-x^2}}. \quad \text{… (ii)}$$

It is being given that when $x = 0$, then $y = 2$.

Putting $x = 0$ and $y = 2$ in (ii), we get $C = 2$.

Hence, $y = \frac{-x}{2} + \frac{\sin^{-1} x}{2\sqrt{1-x^2}} + \frac{2}{\sqrt{1-x^2}}$ is the required solution.

EXAMPLE 20 *Find the equation of a curve passing through the point (0, 1), it being given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the x-coordinate and the product of the x-coordinate and y-coordinate of the point.*

SOLUTION We know that the slope of the tangent to the curve is $\frac{dy}{dx}$.

$$\therefore \quad \frac{dy}{dx} = x + xy \;\Rightarrow\; \frac{dy}{dx} - xy = x. \qquad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = -x$ and $Q = x$.

So, the given differential equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{\int -x\,dx} = e^{\frac{-x^2}{2}}.$$

Hence, the solution of the given differential equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\,dx + C,$$

$$\text{i.e., } y \times e^{\frac{-x^2}{2}} = \int x e^{\frac{-x^2}{2}}\,dx + C$$

$$= \int e^{-t}\,dt + C, \text{ where } \frac{x^2}{2} = t$$

$$= -e^{-t} + C = -e^{\frac{-x^2}{2}} + C.$$

$$\therefore \quad y = -1 + Ce^{\frac{x^2}{2}}. \qquad \ldots \text{(ii)}$$

We have to find a curve satisfying (ii) and passing through (0, 1).

Putting $x = 0$ and $y = 1$ in (ii), we get $C = 2$.

Hence, $y = -1 + 2e^{\frac{x^2}{2}}$ is the equation of the required curve.

EXAMPLE 21 *An equation relating to the stability of an aeroplane is given by* $\frac{dv}{dt} = g\cos\alpha - kv$, *where v is the velocity and g, α, k are constants. Find an expression for the velocity if* $v = 0$ *at* $t = 0$.

SOLUTION The given differential equation is

$$\frac{dv}{dt} + kv = g\cos\alpha. \qquad \ldots \text{(i)}$$

This is of the form $\frac{dv}{dt} + Pv = Q$, where $P = k$ and $Q = g\cos\alpha$.

Thus, the given equation is linear.

$\text{IF} = e^{\int P\,dt} = e^{\int k\,dt} = e^{kt}$.

So, the solution of the given differential equation is

$v \times \text{IF} = \int \{Q \times \text{IF}\}\,dt + C,$

i.e., $ve^{kt} = \int (g\cos\alpha)\,e^{kt}\,dt + C$

$= \dfrac{(g\cos\alpha)e^{kt}}{k} + C.$ … (ii)

Now, it is given that $v = 0$ when $t = 0$.

Putting $t = 0$ and $v = 0$ in (ii), we get $C = \dfrac{-g\cos\alpha}{k}$.

$\therefore \quad ve^{kt} = \dfrac{(g\cos\alpha)e^{kt}}{k} - \dfrac{g\cos\alpha}{k}$

$\Rightarrow \quad v = \dfrac{1}{k}(g\cos\alpha)(1 - e^{-kt})$, which is the required expression.

SOLUTION OF $\dfrac{dx}{dy} + Px = Q$

Working Rule for Solving $\dfrac{dx}{dy} + Px = Q$

(i) Find $\text{IF} = e^{\int P\,dy}$.

(ii) The solution is given by $x \times \text{IF} = \int \{Q \times \text{IF}\}\,dy + C.$

EXAMPLE 22 *Find the general solution of the differential equation* $(x + 2y^3)\dfrac{dy}{dx} = y,\ y \neq 0.$

SOLUTION The given differential equation may be written as

$y\dfrac{dx}{dy} = x + 2y^3$

$\Rightarrow \quad \dfrac{dx}{dy} - \dfrac{1}{y}\cdot x = 2y^2.$ … (i)

This is of the form $\dfrac{dx}{dy} + Px = Q$, where $P = -\dfrac{1}{y}$ and $Q = 2y^2$.

Thus, the given equation is linear.

$\text{IF} = e^{\int P\,dy} = e^{\int -\frac{1}{y}\,dy} = e^{-\log y} = e^{\log(y^{-1})} = y^{-1} = \dfrac{1}{y}.$

So, the required solution is

$x \times \text{IF} = \int \{Q \times \text{IF}\}\,dy + C,$

i.e., $x \times \frac{1}{y} = \int \left(2y^2 \times \frac{1}{y} \right) dy + C$

$= \int 2y\, dy + C = y^2 + C.$

Hence, $x = y^3 + Cy$ is the required solution.

EXAMPLE 23 *Find the particular solution of the differential equation* $(\tan^{-1} y - x)dy = (1 + y^2)dx$, *given that when* $x = 0$, $y = 0$.

[CBSE 2009, '13]

SOLUTION The given differential equation may be written as

$$\frac{dx}{dy} = \frac{(\tan^{-1} y - x)}{(1 + y^2)}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{1}{(1 + y^2)} \cdot x = \frac{\tan^{-1} y}{(1 + y^2)}. \qquad \ldots \text{(i)}$$

This is of the form $\frac{dx}{dy} + Px = Q$, where $P = \frac{1}{(1 + y^2)}$ and $Q = \frac{\tan^{-1} y}{(1 + y^2)}$.

Thus, the given differential equation is linear.

$\text{IF} = e^{\int P\, dy} = e^{\int \frac{1}{(1+y^2)} dy} = e^{\tan^{-1} y}$.

So, the solution of the given differential equation is given by

$x \times \text{IF} = \int \{Q \times \text{IF}\}\, dy + C$, where C is an arbitrary constant,

i.e., $x \times e^{\tan^{-1} y} = \int \left\{ \frac{\tan^{-1} y}{(1 + y^2)} \times e^{\tan^{-1} y} \right\} dy + C$

$= \int \underset{\text{I}}{t} \underset{\text{II}}{e^t}\, dt + C$, where $\tan^{-1} y = t$ and $\frac{1}{(1 + y^2)} dy = dt$

$= te^t - \int 1 \cdot e^t dt + C$ [integrating by parts]

$= te^t - e^t + C$

$= (t - 1)e^t + C$

$= (\tan^{-1} y - 1)e^{\tan^{-1} y} + C.$

$\therefore \quad xe^{\tan^{-1} y} = (\tan^{-1} y - 1)e^{\tan^{-1} y} + C. \qquad \ldots \text{(ii)}$

Putting $x = 0$ and $y = 0$ in (ii), we get $C = 1$.

$\therefore \quad xe^{\tan^{-1} y} = (\tan^{-1} y - 1)e^{\tan^{-1} y} + 1.$

Hence, $x = (\tan^{-1} y - 1) + e^{-\tan^{-1} y}$ is the required solution.

EXAMPLE 24 *Find the particular solution of the differential equation* $(\sqrt{1 - y^2})dx = (\sin^{-1} y - x)dy$, *it being given that when* $y = 0$, *then* $x = 0$.

SOLUTION The given differential equation may be written as

$$\frac{dx}{dy} = \frac{\sin^{-1} y - x}{\sqrt{1 - y^2}}$$

$$\Rightarrow \quad \frac{dx}{dy}+\frac{1}{\sqrt{1-y^2}}\cdot x=\frac{\sin^{-1}y}{\sqrt{1-y^2}}. \qquad \ldots \text{(i)}$$

This is of the form $\frac{dx}{dy}+Px=Q$, where $P=\frac{1}{\sqrt{1-y^2}}$ and $Q=\frac{\sin^{-1}y}{\sqrt{1-y^2}}$.

Thus, the given equation is linear.

$$\text{IF}=e^{\int P\,dy}=e^{\int\frac{1}{\sqrt{1-y2}}dy}=e^{\sin^{-1}y}.$$

So, the solution of the given differential equation is given by

$$x\times\text{IF}=\int(Q\times\text{IF})\,dy+C,$$

$$\text{i.e., } x\times e^{\sin^{-1}y}=\int\left\{\frac{\sin^{-1}y}{\sqrt{1-y^2}}\cdot e^{\sin^{-1}y}\right\}dy+C$$

$$=\int \underset{\text{I}}{t}\,\underset{\text{II}}{e^t}\,dt+C, \text{ where } \sin^{-1}y=t \text{ and } \frac{1}{\sqrt{1-y^2}}dy=dt$$

$$=te^t-\int 1\cdot e^t\,dt+C$$

$$=te^t-\int e^t\,dt+C=(te^t-e^t)+C$$

$$=(t-1)e^t+C$$

$$=(\sin^{-1}y-1)e^{\sin^{-1}y}+C \qquad [\because \quad t=\sin^{-1}y].$$

$$\therefore \quad x=(\sin^{-1}y-1)+Ce^{-\sin^{-1}y}. \qquad \ldots \text{(ii)}$$

Putting $y=0$ and $x=0$ in (ii), we get $C=1$.

Hence, $x=(\sin^{-1}y-1)+e^{-\sin^{-1}y}$ is the required solution.

EXAMPLE 25 *Find the particular solution of the differential equation* $\frac{dx}{dy}+x\cot y=2y+y^2\cot y\ (y\neq 0)$, *given that* $x=0$ *when* $y=\frac{\pi}{2}$.

[CBSE 2013C]

SOLUTION The given differential equation is of the form $\frac{dx}{dy}+Px=Q$, where $P=\cot y$ and $Q=(2y+y^2\cot y)$.

So, the given differential equation is linear.

$$\text{IF}=e^{\int P\,dy}=e^{\int\cot y\,dy}=e^{\log\sin y}=\sin y.$$

Therefore, the solution is given by

$$x\times\text{IF}=\int(Q\times\text{IF})\,dy+C,$$

$$\text{i.e., } x\times\sin y=\int(2y+y^2\cot y)\sin y\,dy+C$$

$$=\int 2y\sin y\,dy+\int \underset{\text{I}}{y^2}\underset{\text{II}}{\cos y}\,dy+C$$

$$= \int 2y \sin y\, dy + [y^2 \sin y - \int 2y \sin y\, dy] + C$$

$$= y^2 \sin y + C.$$

$\therefore \quad x = y^2 + C \operatorname{cosec} y.$ … (i)

Putting $y = \frac{\pi}{2}$ and $x = 0$ in (i), we get $C = \frac{-\pi^2}{4}$.

Hence, the required solution is $x = y^2 - \frac{\pi^2}{4}\operatorname{cosec} y$.

EXERCISE 21

Find the general solution for each of the following differential equations.

1. $\frac{dy}{dx} + \frac{1}{x} \cdot y = x^2$
2. $x\frac{dy}{dx} + 2y = x^2$
3. $2x\frac{dy}{dx} + y = 6x^3$ **[CBSE 2003C]**
4. $x\frac{dy}{dx} + y = 3x^2 - 2,\ x > 0$
5. $x\frac{dy}{dx} - y = 2x^3$ **[CBSE 2006]**
6. $x\frac{dy}{dx} - y = x + 1$
7. $(1 + x^2)\frac{dy}{dx} + 2xy = \frac{1}{(1 + x^2)}$
8. $(1 - x^2)\frac{dy}{dx} + xy = x\sqrt{1 - x^2}$
9. $(1 - x^2)\frac{dy}{dx} + xy = ax$ **[CBSE 2006C]**
10. $(x^2 + 1)\frac{dy}{dx} - 2xy = (x^2 + 1)(x^2 + 2)$
11. $\frac{dy}{dx} + 2y = 6e^x$ **[CBSE 2007]**
12. $\frac{dy}{dx} + 3y = e^{-2x}$
13. $\frac{dy}{dx} + 8y = 5e^{-3x}$ **[CBSE 2007]**
14. $x\frac{dy}{dx} - y = (x - 1)e^x,\ x > 0$
15. $\frac{dy}{dx} - y \tan x = e^x \sec x$
16. $(x \log x)\frac{dy}{dx} + y = 2\log x$ **[CBSE 2009]**
17. $x\frac{dy}{dx} + y = x \log x$ **[CBSE 2008]**
18. $x\frac{dy}{dx} + 2y = x^2 \log x$
19. $(1 + x)\frac{dy}{dx} - y = e^{3x}(1 + x)^2$ **[CBSE 2008]**
20. $\frac{dy}{dx} + \frac{4x}{(x^2 + 1)}y + \frac{1}{(x^2 + 1)^2} = 0$
21. $(y + 3x^2)\frac{dx}{dy} = x$ **[CBSE 2011]**
22. $x\,dy - (y + 2x^2)dx = 0$ **[CBSE 2011]**
23. $x\,dy + (y - x^3)dx = 0$ **[CBSE 2011]**
24. $\frac{dy}{dx} + 2y = \sin x$
25. $\frac{dy}{dx} + y = \cos x - \sin x$ **[CBSE 2009]**
26. $\sec x\frac{dy}{dx} - y = \sin x$ **[CBSE 2009C]**
27. $(1 + x^2)dy + 2xy\,dx = \cot x$ **[CBSE 2011C]**
28. $(\sin x)\frac{dy}{dx} + (\cos x)y = \cos x \sin^2 x$
29. $\frac{dy}{dx} + 2y \cot x = 3x^2 \operatorname{cosec}^2 x$

30. $x\frac{dy}{dx} - y = 2x^2 \sec x$

31. $\frac{dy}{dx} = y \tan x - 2\sin x$

32. $\frac{dy}{dx} + y \cot x = \sin 2x$ **[CBSE 2008]**

33. $\frac{dy}{dx} + 2y \tan x = \sin x$ **[CBSE 2008]**

34. $\frac{dy}{dx} + y \cot x = x^2 \cot x + 2x$ **[CBSE 2005]**

Find a particular solution satisfying the given condition for each of the following differential equations.

35. $x\frac{dy}{dx} + y = x^3$, given that $y = 1$ when $x = 2$.

36. $\frac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x$, given that $y = 0$ when $x = \frac{\pi}{2}$. **[CBSE 2012]**

37. $\frac{dy}{dx} + 2xy = x$, given that $y = 1$ when $x = 0$.

38. $\frac{dy}{dx} + 2y = e^{-2x} \sin x$, given that $y = 0$ when $x = 0$.

39. $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$, given that $y = 0$ when $x = 0$.

40. $x\frac{dy}{dx} - y = \log x$, given that $y = 0$ when $x = 1$.

41. $\frac{dy}{dx} + y \tan x = 2x + x^2 \tan x$, given that $y = 1$ when $x = 0$.

42. A curve passes through the origin and the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point. Find the equation of the curve.

43. A curve passes through the point (0, 2) and the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5. Find the equation of the curve.

Find the general solution for each of the following differential equations.

44. $ydx - (x + 2y^2)dy = 0$

45. $ydx + (x - y^2)dy = 0$

46. $(x - y^3)\frac{dy}{dx} + y = 0$ **[CBSE 2011]**

47. $(x + 3y^2)\frac{dy}{dx} = y, (y > 0)$

48. $(x + y)\frac{dy}{dx} = 1$

49. $(x + y + 1)\frac{dy}{dx} = 1$

50. Solve $(x + 1)\frac{dy}{dx} = 2e^{-y} - 1$, given that $x = 0$ when $y = 0$.

51. Solve $(1 + y^2)dx + (x - e^{-\tan^{-1} y})dy = 0$, given that when $y = 0$, then $x = 0$.

ANSWERS (EXERCISE 21)

1. $y = \frac{x^3}{4} + \frac{C}{x}$ **2.** $y = \frac{x^2}{4} + \frac{C}{x^2}$ **3.** $y = \frac{6}{7}x^3 + \frac{C}{\sqrt{x}}$ **4.** $y = x^2 - 2 + \frac{C}{x}$

5. $y = x^3 + Cx$ **6.** $y = x \log\ x - 1 + Cx$ **7.** $y(1+x^2) = \tan^{-1}x + C$

8. $y = \frac{-1}{2}\sqrt{1-x^2}\log\,(1-x^2) + C\sqrt{1-x^2}$ **9.** $y = a + C\sqrt{1-x^2}$

10. $y = (x^2+1)(x + \tan^{-1}x + C)$ **11.** $y = 2e^x + Ce^{-2x}$ **12.** $y = e^{-2x} + Ce^{-3x}$

13. $y = \frac{-5}{4}e^{-3x} + Ce^{-2x}$ **14.** $y = e^x + Cx$ **15.** $y\cos x = e^x + C$

16. $y(\log\ x) = (\log\ x)^2 + C$ **17.** $4xy = 2x^2\log\ x - x^2 + C$

18. $y = \frac{x^2}{16}(4\log\ x - 1) + \frac{C}{x^2}$ **19.** $y = \frac{1}{3}(1+x)e^{3x} + C(1+x)$

20. $y = \frac{-x}{(x^2+1)^2} + \frac{C}{(x^2+1)^2}$ **21.** $y = 3x^2 + Cx$ **22.** $y = 2x^2 + Cx$

23. $y = \frac{x^3}{4} + \frac{C}{x}$ **24.** $y = \frac{1}{5}(2\sin\ x - \cos\ x) + Ce^{-2x}$ **25.** $y = \cos\ x + Ce^{-x}$

26. $y = Ce^{\sin\ x} - (1 + \sin\ x)$ **27.** $y(1+x^2) = \log\,|\sin\ x| + C$

28. $y\sin\ x = \frac{1}{3}\sin^3 x + C$ **29.** $y\sin^2 x = x^3 + C$

30. $y = Cx + 2x\log\,|\sec x + \tan\ x|$ **31.** $2y\cos\ x = \cos\ 2x + C$

32. $y\sin\ x = \frac{2}{3}\sin^3 x + C$ **33.** $y = \cos\ x + C\cos^2 x$

34. $y = x^2 + C\,\text{cosec}\ x$ **35.** $y = \frac{x^3}{4} - \frac{2}{x}$ **36.** $y\sin\ x = 2x^2 - \frac{1}{2}\pi^2$

37. $2y = 1 + e^{-x^2}$ **38.** $y = \sin\ x$ **39.** $y = \frac{4x^3}{3(1+x^2)}$ **40.** $y = x - 1 - \log\ x$

41. $y = x^2 + \cos x$ **42.** $x + y + 1 = e^x$ **43.** $y = 4 - x - 2e^x$ **44.** $x = 2y^2 + Cy$

45. $x = \frac{y^2}{3} + \frac{C}{y}$ **46.** $x = \frac{y^3}{4} + \frac{C}{y}$ **47.** $x = 3y^2 + Cy$

48. $(x + y + 1)e^{-y} = C$ **49.** $x = Ce^y - (y+2)$ **50.** $y = \log\left|\frac{2x+1}{x+1}\right|$

51. $xe^{\tan^{-1}y} = \tan^{-1}y$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 21)

8. $\frac{dy}{dx} + \frac{x}{(1-x^2)}\cdot y = \frac{x}{\sqrt{1-x^2}}\cdot$

$\text{IF} = e^{\frac{-1}{2}\int\frac{-2x}{(1-x^2)}dx} = e^{-\frac{1}{2}\log\,(1-x^2)} = e^{\log\frac{1}{\sqrt{1-x^2}}} = \frac{1}{\sqrt{1-x^2}}$

$\therefore$ its solution is

$$y \times \frac{1}{\sqrt{1-x^2}} = \int \frac{x}{\sqrt{1-x^2}} \cdot \frac{1}{\sqrt{1-x^2}}\,dx = \frac{-1}{2}\int \frac{-2x}{(1-x^2)}\,dx + C$$

$$= -\frac{1}{2}\log(1-x^2) + C.$$

9. $\dfrac{dy}{dx} + \dfrac{x}{(1-x^2)} \cdot y = \dfrac{ax}{(1-x^2)}$.

$$\text{IF} = e^{-\frac{1}{2}\int \frac{-2x}{(1-x^2)}\,dx} = e^{-\frac{1}{2}\log(1-x^2)} = \frac{1}{\sqrt{1-x^2}}.$$

$\therefore$ its solution is given by

$$y \times \frac{1}{\sqrt{1-x^2}} = \int \frac{1}{\sqrt{1-x^2}} \times \frac{ax}{(1-x^2)}\,dx + C$$

$$= -\frac{a}{2}\int \frac{-2x}{(1-x^2)^{3/2}}\,dx + C = -\frac{a}{2}\int \frac{dt}{t^{3/2}} + C, \text{ where } (1-x^2) = t$$

$$= -\frac{a}{2}\int t^{-3/2}\,dt + C = \frac{a}{\sqrt{t}} + C = \frac{a}{\sqrt{1-x^2}} + C.$$

10. Write $(x^2 + 2) = \{(x^2 + 1) + 1\}$.

14. $\dfrac{dy}{dx} - \dfrac{1}{x} \cdot y = \dfrac{(x-1)}{x} e^x$.

$$\text{IF} = e^{-\int \frac{1}{x}\,dx} = e^{-\log x} = e^{\log(1/x)} = \frac{1}{x}.$$

$\therefore$ its solution is given by

$$y \times \frac{1}{x} = \int e^x \left(\frac{x-1}{x^2}\right) dx + C = \int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx + C$$

$$= e^x \cdot \frac{1}{x} + C. \quad [\because \int e^x \{f(x) + f'(x)\}\,dx = e^x f(x)].$$

16. $\dfrac{dy}{dx} + \dfrac{1}{x\log x} \cdot y = \dfrac{2}{x}$.

$$\text{IF} = e^{\int \frac{1}{x\log x}\,dx} = e^{\int \frac{1}{t}\,dt} = e^{\log t} = t = \log x.$$

21. $\dfrac{dy}{dx} = \dfrac{y + 3x^2}{x} \Rightarrow \dfrac{dy}{dx} - \dfrac{1}{x} \cdot y = 3x.$

22. $\dfrac{dy}{dx} = \dfrac{y + 2x^2}{x} \Rightarrow \dfrac{dy}{dx} - \dfrac{1}{x} \cdot y = 2x.$

23. $\dfrac{dy}{dx} = \dfrac{x^3 - y}{x} \Rightarrow \dfrac{dy}{dx} + \dfrac{1}{x} \cdot y = x^2.$

24. $\text{IF} = e^{\int 2\,dx} = e^{2x}$

$\therefore$ $y \times e^{2x} = \int e^{2x} \sin x\,dx + C$

$$= e^{2x}\left\{\frac{2\sin x - \cos x}{(2^2 + 1^2)}\right\} + C \quad \left[\int e^{ax} \sin bx\,dx = e^{ax}\left\{\frac{a\sin bx + b\cos bx}{(a^2 + b^2)}\right\}\right]$$

$$\Rightarrow \quad y = \frac{(2\sin x - \cos x)}{5} + Ce^{-2x}.$$

25. $\text{IF} = e^{\int dx} = e^x$.

$y \times e^x = \int e^x(\cos x - \sin x)\,dx + C$

$= \int \underset{\text{I}}{(\cos x)}\,\underset{\text{II}}{e^x}\,dx - \int (\sin x)e^x dx + C$

$= (\cos x)e^x - \int (-\sin x)e^x dx - \int (\sin x)e^x dx + C$

$= (\cos x)e^x + C$.

26. $\dfrac{dy}{dx} - (\cos x)y = \sin x \cos x$.

$\text{IF} = e^{-\int \cos x\,dx} = e^{-\sin x}$.

$\therefore \quad y \times e^{-\sin x} = \int (\sin x \cos x)\,e^{-\sin x} dx + C$

$= \int \underset{\text{I}}{t}\,\underset{\text{II}}{e^{-t}}\,dt + C$, where $\sin x = t$.

32. $\text{IF} = e^{\int \cot x\,dx} = e^{\log \sin x} = \sin x$.

$\therefore \quad y \times \sin x = \int \sin x \sin 2x\,dx + C$

$= 2\int \sin^2 x \cos x\,dx + C = 2\int t^2 dt + C$, where $\sin x = t$.

33. $\text{IF} = e^{\int 2\tan x\,dx} = e^{-2\log \cos x} = e^{\log (\cos x)^{-2}} = (\cos x)^{-2} = \dfrac{1}{\cos^2 x}$.

$y \times \dfrac{1}{\cos^2 x} = \int \left\{ \sin x \times \dfrac{1}{\cos^2 x} \right\} dx + C = \int \sec x \tan x\,dx + C = \sec x + C$

$\therefore \quad y = \cos x + C \cos^2 x$.

34. $\text{IF} = e^{\int \cot x\,dx} = e^{\log \sin x} = \sin x$.

$y \times \sin x = \int (x^2 \cot x + 2x) \sin x\,dx + C$

$= \int x^2 \cos x\,dx + \int 2x \sin x\,dx + C$

$= x^2 \sin x - \int 2x \sin x\,dx + \int 2x \sin x\,dx + C = x^2 \sin x + C$.

37. $\text{IF} = e^{\int 2x\,dx} = e^{x^2}$.

$\therefore \quad y \times e^{x^2} = \int x e^{x^2} dx = \dfrac{1}{2}\int e^t dt + C$, where $x^2 = t$

$= \dfrac{1}{2} e^{x^2} + C$.

$\therefore \quad y = \dfrac{1}{2} + \dfrac{C}{e^{x^2}}$.

Putting $y = 1$ and $x = 0$, we get $C = \dfrac{1}{2}$.

40. $\text{IF} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = e^{\log (1/x)} = \dfrac{1}{x}$.

$y \times \dfrac{1}{x} = \int \left(\underset{\text{I}}{\log x} \cdot \underset{\text{II}}{\dfrac{1}{x^2}} \right) dx + C$.

43. $(x + y) - \dfrac{dy}{dx} = 5$.

44. $y\,dx = (x + 2y^2)dy \Rightarrow \dfrac{dx}{dy} = \dfrac{x}{y} + 2y$

$\therefore \quad \dfrac{dx}{dy} - \dfrac{1}{y} \cdot x = 2y$.

45. $y\,dx = (y^2 - x)dy \Rightarrow \dfrac{dx}{dy} = \dfrac{y^2 - x}{y}$

$\therefore \quad \dfrac{dx}{dy} - \dfrac{1}{y} \cdot x = y.$

46. $(y^3 - x)\dfrac{dy}{dx} = y \Rightarrow \dfrac{dx}{dy} = \dfrac{(y^3 - x)}{y}$

$\therefore \quad \dfrac{dx}{dy} + \dfrac{1}{y} \cdot x = y^2.$

47. $(x + 3y^2)\dfrac{dy}{dx} = y \Rightarrow \dfrac{dx}{dy} = \dfrac{x + 3y^2}{y}$

$\therefore \quad \dfrac{dx}{dy} - \dfrac{1}{y} \cdot x = 3y.$

48. $(x + y)\dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dx}{dy} = x + y.$

$\therefore \quad \dfrac{dx}{dy} - x = y.$

$\text{IF} = e^{-\int dy} = e^{-y}.$

$\therefore \quad x \times e^{-y} = \int \underset{\text{I}}{y}\, \underset{\text{II}}{e^{-y}}\, dy + C.$

49. $\dfrac{dx}{dy} - x = (y + 1).$

$\text{IF} = e^{-\int dy} = e^{-y}.$

$x \times e^{-y} = \int (y + 1)e^{-y} dy + C$

$= \int \underset{\text{I}}{y}\, \underset{\text{II}}{e^{-y}}\, dy + \int e^{-y} dy + C$

$= -ye^{-y} + 2\int e^{-y} dy + C$

$= -ye^{-y} - 2e^{-y} + C.$

$\therefore \quad x = Ce^{y} - (y + 2).$

50. $(x + 1)\dfrac{dy}{dx} = \dfrac{(2 - e^y)}{e^y} \Rightarrow \dfrac{dx}{dy} = \dfrac{(x + 1)e^y}{(2 - e^y)}$

$\therefore \quad \dfrac{dx}{dy} - \dfrac{e^y}{(2 - e^y)} \cdot x = \dfrac{e^y}{(2 - e^y)}.$

$\text{IF} = e^{\int \frac{-e^y}{(2-e^y)} dy} = e^{\log (2-e^y)} = (2 - e^y).$

$\therefore \quad x \times (2 - e^y) = \int \dfrac{e^y}{(2 - e^y)} \times (2 - e^y) dy + C = \int e^y dy + C = e^y + C.$

Now, $y = 0$ and $x = 0 \Rightarrow C = -1.$

$\therefore \quad x(2 - e^y) = e^y - 1 \Rightarrow (2x + 1) = (x + 1)e^y$

$\therefore \quad e^y = \dfrac{(2x + 1)}{(x + 1)} \Rightarrow y = \log\left(\dfrac{2x + 1}{x + 1}\right).$

51. $(1+y^2)dx = (e^{-\tan^{-1} y} - x)dy \Rightarrow \dfrac{dx}{dy} = \dfrac{e^{-\tan^{-1} y} - x}{(1+y^2)}$.

$\therefore \quad \dfrac{dx}{dy} + \dfrac{1}{(1+y^2)} \cdot x = \dfrac{e^{-\tan^{-1} y}}{(1+y^2)}$.

$\text{IF} = e^{\int \frac{1}{(1+y^2)} dy} = e^{\tan^{-1} y}$.

$\therefore \quad x \times e^{\tan^{-1} y} = \int \left\{ \dfrac{e^{-\tan^{-1} y}}{(1+y^2)} \times e^{\tan^{-1} y} \right\} dy = C = \int \dfrac{1}{(1+y^2)} dy + C$

$\Rightarrow \quad x \times e^{\tan^{-1} y} = \tan^{-1} y + C.$ … (i)

Putting $x = 0$ and $y = 0$ in (i), we get $C = 0$.

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. The solution of the DE $\dfrac{dy}{dx} = e^{x+y}$ is

   (a) $e^x + e^y = C$ (b) $e^x - e^{-y} = C$ (c) $e^x + e^{-y} = C$ (d) none of these

2. The solution of the DE $\dfrac{dy}{dx} = 2^{x+y}$ is

   (a) $2^x + 2^y = C$ (b) $2^x + 2^{-y} = C$ (c) $2^x - 2^{-y} = C$ (d) none of these

3. The solution of the DE $(e^x + 1)y\,dy = (y+1)e^x dx$ is

   (a) $e^y = C(e^x + 1)(y+1)$ (b) $e^y = e^x + y + 1$

   (c) $y = (e^x + 1)(y+1)$ (d) none of these

4. The solution of the DE $x\,dy + y\,dx = 0$ is

   (a) $x + y = C$ (b) $xy = C$ (c) $\log(x+y) = C$ (d) none of these

5. The solution of the DE $x\dfrac{dy}{dx} = \cot y$ is

   (a) $x \cos y = C$ (b) $x \tan y = C$ (c) $x \sec y = C$ (d) none of these

6. The solution of the DE $\dfrac{dy}{dx} = \dfrac{(1+y^2)}{(1+x^2)}$ is

   (a) $(y+x) = C(1-yx)$ (b) $(y-x) = C(1+yx)$

   (c) $y = (1+x)C$ (d) none of these

7. The solution of the DE $\dfrac{dy}{dx} = 1 - x + y - xy$ is

   (a) $\log(1+y) = x - \dfrac{x^2}{2} + C$ (b) $e^{(1+y)} = x - \dfrac{x^2}{2} + C$

   (c) $e^y = x - \dfrac{x^2}{2} + C$ (d) none of these

8. The solution of the DE $\frac{dy}{dx} = e^{x+y} + x^2 \cdot e^y$ is

(a) $e^{x-y} + \frac{x^3}{3} + C$ (b) $e^x + e^{-y} + \frac{x^3}{3} = C'$

(c) $e^x - e^{-y} = \frac{x^3}{3} + C$ (d) none of these

9. The solution of the DE $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$ is

(a) $y + \sin^{-1} y = \sin^{-1} x + C$ (b) $\sin^{-1} y - \sin^{-1} x = C$

(c) $\sin^{-1} y + \sin^{-1} x = C$ (d) none of these

10. The solution of the DE $\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$ is

(a) $y = 2\tan\frac{x}{2} - x + C$ (b) $y = \tan\frac{x}{2} - 2x + C$

(c) $y = \tan x - x + C$ (d) none of these

11. The solution of the DE $\frac{dy}{dx} = \frac{-2xy}{(x^2+1)}$ is

(a) $y^2(x+1) = C$ (b) $y(x^2+1) = C$ (c) $x^2(y+1) = C$ (d) none of these

12. The solution of the DE $\cos x(1+\cos y)dx - \sin y(1+\sin x)dy = 0$ is

(a) $1 + \sin x \cos y = C$ (b) $(1+\sin x)(1+\cos y) = C$

(c) $\sin x \cos y + \cos x = C$ (d) none of these

13. The solution of the DE $x\cos y\, dy = (xe^x \log x + e^x)dx$ is

(a) $\sin y = e^x + \log x + C$ (b) $\sin y - e^x + \log x = C$

(c) $\sin y = e^x(\log x) + C$ (d) none of these

14. The solution of the DE $\frac{dy}{dx} + y \log y \cot x = 0$ is

(a) $\cos x \log y = C$ (b) $\sin x \log y = C$

(c) $\log y = C\sin x$ (d) none of these

15. The general solution of the DE $(1+x^2)dy - xy\,dx = 0$ is

(a) $y = C(1+x^2)$ (b) $y^2 = C(1+x^2)$ (c) $y\sqrt{1+x^2} = C$ (d) none of these

16. The general solution of the DE $x\sqrt{1+y^2}\,dx + y\sqrt{1+x^2}dy = 0$ is

(a) $\sin^{-1} x + \sin^{-1} y = C$ (b) $\sqrt{1+x^2} + \sqrt{1+y^2} = C$

(c) $\tan^{-1} x + \tan^{-1} y = C$ (d) none of these

17. The general solution of the DE $\log\left(\frac{dy}{dx}\right) = (ax + by)$ is

(a) $\frac{-e^{-by}}{b} = \frac{e^{ax}}{a} + C$ (b) $e^{ax} - e^{-by} = C$

(c) $be^{ax} + ae^{by} = C$ (d) none of these

18. The general solution of the DE $\frac{dy}{dx} = (\sqrt{1-x^2})(\sqrt{1-y^2})$ is

(a) $\sin^{-1} y - \sin^{-1} x = x\sqrt{1-x^2} + C$ (b) $2\sin^{-1} y - \sin^{-1} x = x\sqrt{1-x^2} + C$

(c) $2\sin^{-1} y - \sin^{-1} x = C$ (d) none of these

19. The general solution of the DE $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$ is

(a) $x^2 - y^2 = C_1 x$ (b) $x^2 + y^2 = C_1 y$

(c) $x^2 + y^2 = C_1 x$ (d) none of these

20. The general solution of the DE $x^2 \frac{dy}{dx} = x^2 + xy + y^2$ is

(a) $\tan^{-1} \frac{y}{x} = \log x + C$ (b) $\tan^{-1} \frac{x}{y} = \log x + C$

(c) $\tan^{-1} \frac{y}{x} = \log y + C$ (d) none of these

21. The general solution of the DE $x\frac{dy}{dx} = y + x\tan\frac{y}{x}$ is

(a) $\sin\left(\frac{y}{x}\right) = C$ (b) $\sin\left(\frac{y}{x}\right) = Cx$ (c) $\sin\left(\frac{y}{x}\right) = Cy$ (d) none of these

22. The general solution of the DE $2xy\,dy + (x^2 - y^2)dx = 0$ is

(a) $x^2 + y^2 = Cx$ (b) $x^2 + y^2 = Cy$ (c) $x^2 + y^2 = C$ (d) none of these

23. The general solution of the DE $(x - y)dy + (x + y)dx$ is

(a) $\tan^{-1} \frac{y}{x} = C\sqrt{x^2 + y^2}$ (b) $e^{\tan^{-1}(y/x)} = C\sqrt{x^2 + y^2}$

(c) $\tan^{-1}\left(\frac{y}{x}\right) = x^2 + y^2 + C$ (d) none of these

24. The general solution of the DE $\frac{dy}{dx} = \frac{y}{x} + \sin\frac{y}{x}$ is

(a) $\tan\frac{y}{2x} = Cx$ (b) $\tan\frac{y}{x} = Cx$ (c) $\tan\frac{y}{2x} = C$ (d) none of these

25. The general solution of the DE $\frac{dy}{dx} + y\tan x = \sec x$ is

(a) $y = \sin x - C\cos x$ (b) $y = \sin x + C\cos x$

(c) $y = \cos x - C\sin x$ (d) none of these

26. The general solution of the DE $\frac{dy}{dx} + y \cot x = 2\cos x$ is

(a) $(y + \sin x)\sin x = C$ (b) $(y + \cos x)\sin x = C$
(c) $(y - \sin x)\sin x = C$ (d) none of these

27. The general solution of the DE $\frac{dy}{dx} + \frac{y}{x} = x^2$ is

(a) $xy = x^4 + C$ (b) $4xy = x^4 + C$ (c) $3xy = x^3 + C$ (d) none of these

ANSWERS (OBJECTIVE QUESTIONS)

1. (c)	**2.** (b)	**3.** (a)	**4.** (b)	**5.** (a)	**6.** (b)	**7.** (a)	**8.** (b)	**9.** (c)
10. (a)	**11.** (b)	**12.** (b)	**13.** (c)	**14.** (b)	**15.** (b)	**16.** (b)	**17.** (a)	**18.** (b)
19. (c)	**20.** (a)	**21.** (b)	**22.** (a)	**23.** (b)	**24.** (a)	**25.** (b)	**26.** (c)	**27.** (b)

HINTS TO THE GIVEN OBJECTIVE QUESTIONS

1. The given DE is $e^x dx = e^{-y} dy$.

$\therefore \int e^x dx = \int e^{-y} dy \Rightarrow e^x = -e^{-y} + C \Rightarrow e^x + e^{-y} = C.$

2. We have $2^x dx = 2^{-y} dy \Rightarrow \int 2^x dx = \int 2^{-y} dy.$

$\therefore \quad 2^x \log 2 = -2^{-y} \log 2 = \log C \Rightarrow 2^x + 2^{-y} = C$, where $\frac{\log C}{\log 2} = C.$

3. $\int \frac{e^x}{(e^x + 1)} dx = \int \left(\frac{y}{y+1} \right) dy \Rightarrow \log (e^x + 1) = \int \left(1 - \frac{1}{y+1} \right) dy$

$\Rightarrow \quad y - \log (y + 1) = \log (e^x + 1) + \log C \Rightarrow y = \log [C(e^x + 1)(y + 1)]$

$\Rightarrow \quad e^y = C(e^x + 1)(y + 1).$

4. $x\,dy + y\,dx = 0 \Rightarrow \int \frac{1}{y} dy = -\int \frac{1}{x} dx \Rightarrow \log y = -\log x + \log C.$

$\therefore \quad \log xy = \log C \Rightarrow xy = C.$

5. $\int \tan y\, dy = \int \frac{1}{x} dx \Rightarrow \log x = \log \sec y + \log C.$

$\Rightarrow \quad x = C \sec y \Rightarrow x \cos y = C.$

6. $\int \frac{dy}{(1 + y^2)} = \int \frac{dx}{(1 + x^2)} \Rightarrow \tan^{-1} y = \tan^{-1} x + C_1.$

$\Rightarrow \quad \tan^{-1} y - \tan^{-1} x = C_1 \Rightarrow \tan^{-1} \left(\frac{y - x}{1 + yx} \right) = C_1 \Rightarrow \frac{y - x}{(1 + yx)} = \tan C_1 = C.$

7. $\int \frac{dy}{(1 + y)} = \int (1 - x) dx \Rightarrow \log (1 + y) = x - \frac{x^2}{2} + C.$

8. $\frac{dy}{dx} = e^y (e^x + x^2) \Rightarrow \int (e^x + x^2) dx = \int e^{-y} dy.$

$\Rightarrow \quad -e^{-y} = e^x + \frac{x^3}{3} + C \Rightarrow e^x + e^{-y} + \frac{x^3}{3} = C'.$

9. $\int \frac{1}{\sqrt{1-y^2}}dy + \int \frac{1}{\sqrt{1-x^2}}dx = C \Rightarrow \sin^{-1} y + \sin^{-1} x = C.$

10. $\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x} = \frac{2\sin^2(x/2)}{2\cos^2(x/2)} = \tan^2\frac{x}{2} = \left(\sec^2\frac{x}{2} - 1\right)$

$\Rightarrow \int dy = \int \left(\sec^2\frac{x}{2} - 1\right)dx \Rightarrow y = 2\tan\frac{x}{2} - x + C.$

11. $\int \frac{1}{y}dy = \int \frac{-2x}{(x^2+1)}dx \Rightarrow \log y + \log(x^2+1) = \log C.$

$\therefore \quad y(x^2+1) = C.$

12. $\int \frac{\cos x}{(1+\sin x)}dx - \int \frac{\sin y}{(1+\cos y)}dy = \log C$

$\Rightarrow \log(1+\sin x) + \log(1+\cos y) = \log C \Rightarrow (1+\sin x)(1+\cos y) = C.$

13. $\int \cos y\, dy = \int e^x\left(\log x + \frac{1}{x}\right)dx \Rightarrow \sin y = e^x \log x + C.$

14. $\frac{1}{y\log y}dy + \cot x\, dx = 0 \Rightarrow \frac{dy}{y\log y} + \int \cot x\, dx = \log C$

$\Rightarrow \log(\log y) + \log\sin x = \log C \Rightarrow \log(\sin x + \log y) = \log C$

$\Rightarrow (\sin x)\log y = C.$

15. $\int \frac{1}{y}dy = \frac{1}{2}\int \frac{2x}{(1+x^2)}dx \Rightarrow \log y = \frac{1}{2}\log(1+x^2) + \log C$

$\Rightarrow \log y = \log(C\sqrt{1+x^2}) \Rightarrow y = C\sqrt{1+x^2}.$

16. $\int \frac{2x}{2\cdot\sqrt{1+x^2}}dx + \int \frac{2y}{2\cdot\sqrt{1+y^2}} = C \Rightarrow \frac{1}{2}\int \frac{dt}{\sqrt{t}} + \frac{1}{2}\int \frac{du}{\sqrt{u}} = C,$

where $1+x^2 = t$ and $1+y^2 = u$

$\Rightarrow \sqrt{t} + \sqrt{u} = C \Rightarrow \sqrt{1+x^2} + \sqrt{1+y^2} = C.$

17. $\frac{dy}{dx} = e^{ax\,|\,by} \Rightarrow \int e^{\ by}dy = \int e^{ax}dx \Rightarrow \frac{e^{-by}}{-b} = \frac{e^{ax}}{a} + C.$

18. $\int \frac{dy}{\sqrt{1-y^2}} = \int \sqrt{1-x^2}\,dx \Rightarrow \sin^{-1} y = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x + C$

$\Rightarrow 2\sin^{-1} y - \sin^{-1} x = x\sqrt{1-x^2} + C.$

19. The given DE is homogeneous. Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}.$

$\therefore \int \frac{2v}{(1+v^2)}dv = -\int \frac{1}{x}dx \Rightarrow \log(1+v^2) + \log x = \log C.$

$\therefore \quad x^2 + y^2 = C_1 x.$

20. $\frac{dy}{dx} = d\frac{x^2+xy+y^2}{x^2}$, which is homogeneous.

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ to get $v + x\frac{dv}{dx} = (1+v+v^2).$

$\therefore \quad \int \frac{dv}{(1+v^2)} = \int \frac{1}{x}dx \Rightarrow \tan^{-1} v = \log x + C \Rightarrow \tan^{-1}\frac{y}{x} = \log x + C.$

21. $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$, which is homogeneous.

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ to get $\int \cot v\, dv = \int \frac{1}{x} dx$

$\Rightarrow \quad \log \sin v = \log x + \log C \Rightarrow \sin v = Cx \Rightarrow \sin\frac{y}{x} = Cx.$

22. $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$, which is homogeneous.

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Then,

$$\int \frac{2v}{(1+v^2)} dv = -\int \frac{1}{x} dx \Rightarrow \log(1+v^2) = -\log x + \log C$$

$\Rightarrow \quad x(1+v^2) = C \Rightarrow x^2 + y^2 = Cx.$

23. $\frac{dy}{dx} = \frac{x+y}{x-y}$, which is clearly homogeneous.

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx} \Rightarrow \int \frac{(1-v)}{(1+v^2)} dv = \int \frac{1}{x} dx$

$\Rightarrow \quad \int \frac{dv}{(1+v^2)} - \frac{1}{2}\int \frac{2v}{(1+v^2)} dv = \int \frac{1}{x} dx$

$\Rightarrow \quad \tan^{-1} v - \frac{1}{2}\log(1+v^2) = \log x + \log C \Rightarrow \tan^{-1} v = \log\left(Cx\sqrt{1+v^2}\right)$

$\Rightarrow \quad \tan^{-1}\frac{y}{x} = \log\left(C\sqrt{x^2+y^2}\right) \Rightarrow e^{\tan^{-1} y/x} = C\sqrt{x^2+y^2}.$

24. Given DE is homogeneous.

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Then, $v + x\frac{dv}{dx} = v + \sin v \Rightarrow \int \operatorname{cosec} v\, dv = \int \frac{1}{x} dx$

$\Rightarrow \quad \log \tan\frac{v}{2} = \log x + \log C \Rightarrow \tan\frac{v}{2} = Cx \Rightarrow \tan\frac{y}{2x} = Cx.$

25. Given DE is linear.

IF $= e^{\int \tan x\, dx} = e^{\log(\sec x)} = \sec x.$

$\therefore$ its solution is

$y(\sec x) = \int \sec^2 x\, dx \Rightarrow y(\sec x) = \tan x + C \Rightarrow y = \sin x + C\cos x.$

26. Given DE is linear.

IF $= e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x.$

$\therefore \quad y(\sin x) = \int 2\sin x \cos x\, dx \Rightarrow y \sin x = \sin^2 x + C.$

$\therefore \quad (y - \sin x)\sin x = C.$

27. Given DE is linear.

IF $= e^{\int \frac{1}{x} dx} = e^{\log x} = x.$

$\therefore \quad y \times x = \int (x \times x^2)\, dx = \int x^3 dx = \frac{x^4}{4} + C$

$\Rightarrow \quad 4xy = x^4 + C.$